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VALUE, PRICE, AND EXPLOITATION: THE LOGIC OF THE TRANSFORMATION PROBLEM

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Abstract. This paper tries to clarify the logical structure of the relationship between labor values and prices from an axiomatic perspective. The famous "transformation problem" is interpreted as an impossibility result for a specific interpretation of value theory based on specific assumptions and definitions. A comprehensive review of recent literature is provided, which shows that there are various theoretically relevant and logically consistent alternative interpretations based on different assumptions and definitions.

Keywords. Prices; Transformation problem; Value

1. Introduction

When economic activity is organized through markets, the notion of price is central. In the neoclassical individualist tradition, price and value are synonyms; prices emerge in competitive markets from the interaction of optimizing individual households and firms with given endowments, preferences, and technology, and in that sense, the theory of value is subjective. In the Marxian tradition, value is distinct from price and only labor creates value. Such a labor theory of value is objective, and prices are long-run centers of gravity around which market prices fluctuate. The relation between labor value and long-run price is the content of what has become known as the "transformation problem."

The outcome of the last wave of major debates in the 1960s and 1970s has led to the view that the classical-Marxian labor theory of value is logically inconsistent and so irremediably flawed. And even if some of the insights of Marx's theory of value can be salvaged, they are irrelevant for any meaningful positive or normative purposes: as Paul Samuelson (1971) famously put it in his "blackboard theorem," price magnitudes and value magnitudes are independent of each other, with a relation of mutual irrelevance. A similar negative judgment was also shared by commentators who were less hostile to Marxist theory in general. Joan Robinson famously argued, for example, that value theory "provides a typical example of the way metaphysical ideas operate. Logically it is a mere rigmarole of words, but for Marx it was a flood of illumination and for latter-day Marxists, a source of inspiration" (Robinson, 1964, p. 39). Given

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the central role of value theory in Marxian economics, these conclusions have led many commentators to consider the whole of Marx's theory as dead.

This paper argues that such judgments are based on a particular interpretation of Marx's notion of value and of its role in the understanding of capitalist economies. But rather than provide another "solution" to the transformation problem, our aim is to outline a unified theoretical framework that can clarify its logic and in so doing emphasize the wide variety of possible interpretations of value theory for which the transformation problem has no implications. To do this, we employ an axiomatic approach.

Following (Thomson, 2001, p. 332), an axiomatic study has

> the following components: 1. It begins with the specification of a domain of problems, and the formulation of a list of desirable properties of solutions for the domain. 2. It ends with ... descriptions of the families of solutions satisfying various combinations of the properties.

From this perspective, it is possible to conceive of any approach to value theory as (implicitly) defining a set of problems (including definitions of the main variables: prices, values, technology, competition, and so on); formulating a list of desirable properties (axioms) of the labor theory of value, including the specification of the role of value analysis; and then exploring the set of "solutions" to those problems. The axiomatic method then provides a new perspective on some old issues. In particular, the standard approach to the transformation problem—analyzed in Section 5 below—can be seen as identifying an impossibility result: under a certain interpretation of value theory, and on the basis of some axioms concerning value and price magnitudes, the set of solutions is empty. But the axiomatic method also provides a general formal and conceptual framework for analyzing different views on Marxian value theory.

One striking aspect of the debates on Marxian value and price theory is the lack of agreement on virtually anything, including on what the actual issue of contention is. Part of the reason is that the discussion is ideologically loaded. But a large part of the controversy stems from different views concerning the key concepts, the basic methodology, and the actual role of value theory. Different approaches do not simply provide alternative solutions to a given problem, for there is no single way of posing that problem. In terms of Thomson's description above, different approaches often identify a completely different set of problems the solutions to which will have different properties. The advantage of an axiomatic framework is to provide a unified framework to clarify these differences.[1]

There is an enormous literature on the relation between value and price dating back to the publication of *Capital III* in 1894 (Marx, 1981). So, three clarifications concerning the scope of our analysis are in order. First, there are several reviews of earlier debates, including Hunt and Glick (1987), Desai (1988), and Howard and King (1992). This paper focuses only on recent approaches (from around 1980), most of which start from the acknowledgement that, as originally posed, the transformation problem has no solution.[2] Second, we do not provide a survey of all of the (recent) interpretations of Marxian value theory but restrict ourselves to those that have a clear and explicit quantitative dimension. Qualitative interpretations, for example, occasioned by the rediscovery of Rubin (1972), fall outside of our survey. Third, as a logical inquiry into the structure of value theory, we provide no exegetical evidence. All of the approaches we survey can be presented and supported by careful analysis of Marxian texts.

2. What Is Value Theory For?

One of the main arguments of this paper is that disputes on the transformation problem derive from different views about the aim and scope of value theory (in Marx and beyond). And the popular view that Marx's theory is fundamentally flawed derives from the erroneous extrapolation of the logical problems of *one* of the interpretations of value theory to all possible approaches. In this section, we provide a classification and discussion of the main views.

2.1 *Description, Prediction, and Evaluation*

What is the labor theory of value for? The received view is that its main aim is predictive: labor values are meant to explain (relative, equilibrium) prices. Yet, even within a predictive interpretation, labor magnitudes may be relevant to explain other phenomena of capitalist economies. For example, one may argue that the labor theory of value establishes a relation between profits and exploitative relations, thus allowing one to explain investment and growth. More generally, however, it is not clear that Marxian value theory can only be interpreted as a predictive exercise. For "there are at least three distinct non-metaphysical interpretations of the labour theory of value, viz. (i) descriptive, (ii) predictive and (iii) normative" (Sen, 1978, p. 175).

As for (i), Sen (1978, p. 176) notes that "Any description relies on factual statements. But it also involves a selection from the set of factual statements that can be made pertaining to the phenomenon in question: some facts are chosen and others ignored. The selection process is part of the exercise of description, and not a 'metaphysical' exercise." One descriptive interpretation of the labor theory of value is that of capturing the process of formation of equilibrium prices in capitalist economies, as in the standard view. But this is certainly not the only possibility. One may argue that in the labor theory of value, "it is the *activity* of production that is being described, and the selection criterion is focused on 'personal participation'" (Sen, 1978, p. 177). It focuses analysis on human effort and refuses "to give the same status to the ownership of [natural resources and capital] in describing participation in production as personal participation through labour" (Sen, 1978).[3] Thus, alternative formulations of the labor theory of value "have to be judged in terms of the motivation of the exercise of description in the particular case in question" (Sen, 1978, p. 178).

Regarding (iii), the labor theory of value can also be interpreted primarily as providing the foundations for a normative, evaluative exercise and an indictment of capitalist relations of production. For example, one may argue that it explains the origin of profits as accruing from the exploitation of workers and therefore shows the illegitimacy of capitalist earnings, and the source of significant inequalities of well-being. Or it may be taken as providing the foundations of a distributive approach based on contribution and effort.

Thus, even at the most abstract level, there is no single, natural interpretation of the labor theory of value. In addition to the standard predictive view, other interpretations that emphasize its descriptive or evaluative role are possible. These interpretations are not mutually inconsistent. For even within a descriptive approach, the specific interpretation chosen may depend on the actual motivation of the analysis, which can be predictive or evaluative. Recognition of this diversity of interpretation is important when seeking to identify the primary analytical focus of the various approaches: there are a great variety of interpretations of Marxian value theory. Moreover, even within each interpretation (descriptive, predictive, or evaluative), several alternative formulations of the labor theory of value are possible. In the rest of this section, we try to identify some of them.

2.2 *The Equilibrium Price View*

This is the standard approach, and historically the oldest. It contends that Marx's value theory provides an explanation of the equilibrium prices of a competitive market economy. Marx's approach is seen as falling within the classical economists' long-period approach in which profits are interpreted as a surplus. It will be convenient to distinguish two variants, a strong one and a weak one.

Definition 1. According to the *strong equilibrium price view*, relative equilibrium prices are equal to relative values.

Definition 2. According to the *weak equilibrium price view*, relative equilibrium prices are determined by relative values.

The notion of "determination" in Definition 2 can be defined in different ways. Thus, relative equilibrium prices might be determined by relative values in the sense that there exists a clear deterministic functional relation linking the two sets of magnitudes (see, for example, Section 5 below). Alternatively, relative equilibrium prices might be equal to relative values up to a predefined margin of error. We leave the concept of determination undefined in order to accommodate a range of possible views.

2.3 *The Profit & Exploitation View*

In this approach, the basic idea is that the purpose of the labor theory of value is to reveal the origin of profits, the key variable in capitalist economies. At its most basic, capitalist society is a class society of workers and capitalists; these classes exist in antagonistic relation to each other, and that antagonism is based on the extraction of surplus labor from one class (the working class) by the other (the capitalist class). Extraction of surplus labor is called "exploitation," and it characterizes all types of class society. But while exploitation is obvious in, for example, slave societies (slaves are compelled to produce more than they consume) and feudal societies (serfs are compelled to work on the lord's land for part of the week), it is not obvious in capitalist societies where market transactions are voluntary. The purpose of the labor theory of value is then to show how voluntary participation in markets nonetheless generates exploitation. In sum, value theory provides the foundations for the Marxian theory of exploitation, showing that profits result from the exploitation of labor.

Definition 3. According to the *profit & exploitation view*, profits are determined by the exploitation of labor.

2.4 *The Appropriate Level of Analysis*

A further important distinction concerns the appropriate level of value analysis. The standard approach to the labor theory of value is microeconomic and conceives of Marxian value theory as an alternative to Walrasian general equilibrium for the explanation of equilibrium prices. More generally, one can identify a general approach—that can be called, to simplify, the microeconomic view—according to which, value analysis focuses primarily on disaggregated variables, such as prices, or an agent's individual exploitation status, or the labor values of individual commodities.

Definition 4. According to the *microeconomic view*, the labor theory of value applies to disaggregated magnitudes.

Many recent approaches have rejected this microeconomic view, and have interpreted Marxian value theory as explaining primarily macroeconomic features of capitalist economies, such as the aggregate production of surplus value, or the aggregate exploitation rate.

Definition 5. According to the *macroeconomic view*, the labor theory of value is primarily a theory of aggregates.

The different views identified above are not mutually exclusive. An emphasis on aggregate macroeconomic relations as the primary unit of analysis does not mean that microeconomic variables are ignored, or irrelevant. If the equilibrium price view focuses on the relation between labor values and prices, it nevertheless has to incorporate a notion of exploitation for labor values to make sense. So, a value theory that provides an explanation of equilibrium relative prices in capitalist economies must also provide an explanation of equilibrium profits as the product of exploitation. Similarly, if the profit & exploitation view is based on showing how exploitation results from market participation, then some specification of prices is required. Hence, an explanation of profits in terms of exploitation must also provide an account of the prices in which these profits are measured. So, categorizing approaches in the literature to each view is more a matter of determining their emphasis than their exclusive concentration.

Neither are these views exhaustive. But they do encompass the main approaches that are expressed in mathematical language. And given the latter, we emphasize the particular assumptions that drive their different emphases.

3. The Basic Marxian Framework

Marx's vision of production was an advance of capital by a capitalist in order to make profit, an extra sum of money over and above that which the capitalist had to advance to purchase nonlabor means of production, and to purchase the use of labor. These inputs were combined in a production process to produce an output which when sold generated revenues that both replaced the capital advanced and produced more money as profit. The challenge was to find a general explanation for this profit.

For clarity, we set aside the complications related to unproductive labor, fixed capital, international trade, the public sector, joint production, and so on, and consider the simplest production structure, a closed, private economy in which each sector i produces only one type of commodity.[4] There are n such commodities produced in a given production period, using physical inputs in the form of circulating capital and one type of homogeneous labor. Production takes time. Suppose that production processes have a uniform duration and let t denote a generic production period. Let the total value of the output of any sector i during period t, Λ_{it}, be the labor time required to produce the gross output of that sector, Q_{it}. (Unless the context requires it, we will henceforth drop the subscript t.)

Marx argued that what made commodities exchangeable in the market for sums of money, and hence commensurable, was that they were all products of labor. He spent some time refining what he meant by this labor (abstract rather than concrete, simple rather than compound, social rather than private, and necessary rather than wasted), and he measured it in units of socially necessary labor time.[5] Then, the value of a commodity is the sum of the (indirect) labor time embodied in the nonlabor inputs (means of production) and the (direct) labor time expended by workers when using these means of production. Indirect labor time is transferred from the nonlabor means of production to the product of the production process and reappears as part of the value of output. While its location changes, its amount remains the same. For this reason, Marx called the capital advanced to purchase nonlabor means of production "constant capital." When the output is sold, the capitalist recovers the capital he advanced to purchase nonlabor means of production. But he also recovers the capital he advanced to purchase labor inputs, *and* he appropriates a profit. Both of these Marx attributed to the value-creating capacity of human labor: laboring activity uniquely has the ability to produce more than it costs. For this reason, Marx called the capital advanced to purchase labor inputs "variable capital."[6] The extra, over, and above the sum of constant capital and variable capital, Marx called "surplus value." Formally, in labor value terms,

$$\Lambda_i = C_i + V_i + S_i \tag{1}$$

where C_i is total constant capital employed, V_i is total variable capital employed, S_i is total surplus value appropriated by capitalists in sector i, and all magnitudes are measured in units of labor time.

Thus far, these are just definitions. Two steps are required to turn them into a labor *theory* of value. Let the unit value of sector i be given by $\lambda_i = \frac{\Lambda_i}{Q_i}$. While value is measured in units of labor time, that measure is definitional, and it has to be combined with value being measured as a sum of money when the commodity that embodies it is sold. It is this combination that turns a labor *definition* of value into a labor *theory* of value. So, the first step is to assert that if prices were determined by labor values, then at unit level, and for all i,

$$p_i = \frac{\lambda_i}{\lambda_m} \tag{2}$$

where p_i denotes the unit price of commodity i, denominated in units of money; since λ_i is denominated in units of time, the value of money, λ_m, must be denominated in units of time per unit of money.[7]

The second step is to provide some specification of the value of money. For Marx, reflecting the monetary arrangements and institutions of his time, money was a commodity (generally gold); its value was determined, like all commodities, by the labor time required for its production, and its price was unity.[8] Hence, for any value of money, the price of commodity i is determined by its value. Equation (2) specifies the labor theory of value as Marx inherited it from Ricardo (but with more sophisticated notions of labor and labor time).

Three further points concerning equation (2) should be emphasized. First, given the value of money, equation (2) is a statement of "equal" or "equivalent" exchange. Prices exactly reflect values, so that for each and every commodity, its purchaser pays and its seller receives its full value in money terms. That is, for each and every commodity, *its value is exactly conserved in any market transaction*. It will be convenient henceforth to call the prices at which this is true "simple prices."

Second, equation (2) applies to each and every commodity, including labor power (the value-creating capacity of human labor). Applying it to labor power, and denoting its price as w_i, the wage rate per unit of labor time in sector i, and the value of labor-power per unit of labor purchased in sector i as vlp_i, then

$$w_i = \frac{vlp_i}{\lambda_m} \tag{3}$$

As soon as labor mobility is presumed, in long-period equilibrium, the wage rate will be uniform across all sectors, with $w_i = w$, all i, and so, by equation (3), $vlp_i = vlp$, for all sectors i.

Third, suppose that workers do not save and spend their wages on consumer goods (means of subsistence).[9] Let L be the total number of hours worked in the economy. Letting $\mathbf{b} = (b_1, \ldots, b_n)'$ denote the vector indicating the amount of each good i consumed, and denoting whatever the ruling prices are by $\mathbf{p} = (p_1, \ldots, p_n)$, for the economy as a whole $wL = \mathbf{pb}$, so that

$$w = \mathbf{p}\frac{\mathbf{b}}{L} \tag{4}$$

Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$. Combining equations (3) and (4), and assuming equal exchange (equation (2)),

$$vlp = \frac{\boldsymbol{\lambda}\mathbf{b}}{L} \tag{5}$$

or the value of labor power per hour of labor hired is the value of the bundle consumed per hour.[10]

Dividing equation (1) through by Q_i and using equation (2):

$$p_i = \frac{\lambda_i}{\lambda_m} = \left(\frac{C_i}{Q_i} + \frac{V_i}{Q_i} + \frac{S_i}{Q_i}\right)\frac{1}{\lambda_m} = \frac{c_i}{\lambda_m} + \frac{v_i}{\lambda_m} + \frac{s_i}{\lambda_m} \tag{6}$$

By definition, variable capital in sector i is advanced to purchase labor-power in sector i, and in value terms, this must be equal to vlp per hour of labor hired, multiplied by the number of hours purchased l_i. So, $v_i = vlp \cdot l_i$, and since $v_i + s_i = l_i$, then $s_i = (1 - vlp)l_i$. Define the rate of surplus value in sector i as $e_i = s_i/v_i$. Then, $e_i = e_j = e$ for all i and j: because labor mobility implies a uniform wage rate and hence a uniform value of labor power, the rate of surplus value must be uniform across sectors,

$$e = \frac{1 - vlp}{vlp} \tag{7}$$

Thus far, the basic structure of Marx's labor theory of value is the same as Ricardo's: prices are determined by values. And hence, for the same reason as Ricardo discovered (see Sraffa's *Introduction* to Ricardo, 1951), the labor theory of value cannot be true if it is combined with competition. Let the rate of profit earned by capitalists in sector i be given by

$$r_i = \frac{\frac{s_i}{\lambda_m}}{\frac{c_i}{\lambda_m} + \frac{v_i}{\lambda_m}} = \frac{s_i}{c_i + v_i} \tag{8}$$

Marx defined the general rate of profit r^M as

$$r^M = \frac{\frac{\mathbf{sQ}}{\lambda_m}}{\left(\frac{\mathbf{c}}{\lambda_m} + \frac{\mathbf{v}}{\lambda_m}\right)\mathbf{Q}} = \frac{\mathbf{sQ}}{(\mathbf{c}+\mathbf{v})\mathbf{Q}} \tag{9}$$

where $\mathbf{Q}$ is the $n \times 1$ vector of gross output and the (row) value vectors $\mathbf{c}$, $\mathbf{v}$, and $\mathbf{s}$ are defined in similar manner to the vector $\boldsymbol{\lambda}$. In long-period equilibrium, as long as there is capital mobility, competition must equalize the rate of profit across all activities, so that $r_i = r$ for all i. Yet, given equation (7), two competing capitalists who advance identical total quantities of capital, but in different constant and variable amounts, to produce an identical output cannot earn the same rate of profit; more surplus value and hence a higher rate of profit must accrue to the capitalist who advances more variable capital. This is incompatible with capitalist competition, so that, if the rate of profit is to be equalized, value must be transferred from the capitalist who advances more variable capital to the capitalist who advances less.

To be exact, whereas the prices in equations (2) and (6) are simple prices, the prices $\mathbf{p}^M = (p_1^M, \ldots, p_n^M)$ that support an equalized rate of profit are "prices of production," where

$$p_i^M = (1 + r^M)\left(\frac{c_i}{\lambda_m} + \frac{v_i}{\lambda_m}\right) \tag{10}$$

Then, for a given value of money, the capitalist who advances more (less) variable capital must sell his commodity at a price of production less (more) than its simple price. In this manner, competition redistributes value. Hence, as long as proportions of constant and variable capital differ across different production processes, the labor theory of value understood as equation (2) cannot hold, because it is incompatible with capitalist competition.

Differing time structures of embodied labor had bedevilled the Ricardian labor theory of value. Marx translated this into a difference between price of production and simple price on the one hand, and the proportions in which capital was advanced as constant or variable on the other hand. So, define the composition of capital in sector i as k_i where $k_i = \frac{c_i}{v_i}$. Then, combining equations (6) and (7) and rearranging,

$$\lambda_m p_i = (k_i + 1 + e)v_i$$

Similarly, define the whole economy composition of capital k as $k = \frac{\mathbf{cQ}}{\mathbf{vQ}}$. Then, using equation (9), equation (10) can be rewritten as

$$\lambda_m p_i^M = \frac{k + 1 + e}{k + 1}(k_i + 1)v_i$$

On division, and simplifying,

$$\frac{p_i}{p_i^M} = \left(\frac{k+1}{k_i+1}\right) \Big/ \left(\frac{k+1+e}{k_i+1+e}\right)$$

so that

$$k_i \lesseqgtr k \Leftrightarrow p_i \gtreqless p_i^M$$

Marx's prices of production are greater (less) than their corresponding simple prices in those sectors where the sectoral organic composition of capital is higher (lower) than the economy-wide organic composition of capital. But all sectoral transfers of value will cancel out in the aggregate because it is easily checked that $\mathbf{p}^M\mathbf{Q} = \frac{\boldsymbol{\lambda}\mathbf{Q}}{\lambda_m} = \mathbf{pQ}$. Furthermore, it is definitionally true that aggregate profits, $\Pi = r^M(\frac{\mathbf{c}}{\lambda_m} + \frac{\mathbf{v}}{\lambda_m})\mathbf{Q}$, are equal to aggregate surplus value at simple prices $\frac{\mathbf{sQ}}{\lambda_m}$.

In sum, Marx developed Ricardo's labor theory of value not only by refining the concept of labor time but also by transferring the focus away from simple prices. Equation (2) does not, in general, hold, and price-of-production-simple-price deviations are the norm. That is, unequal exchange, or nonequivalent exchange, is the norm. For Marx, what anchored the system to labor values were the two aggregate equalities; first, aggregate gross value is invariant to whether it is measured in simple prices or prices of production, and second, aggregate surplus value measured in simple prices is identical to aggregate profits measured in prices of production. Thus, the proportionality of price and value magnitudes holds only at the aggregate level, and only at that level is there equal or equivalent exchange.

But there are two difficulties with this account. First, if prices of production systematically differ from their corresponding simple prices in a long-period equilibrium, then the right-hand side of equation (10) appears to be wrongly specified. In a long-period equilibrium, inputs must be evaluated at prices of production, not simple prices. Hence Marx's procedure seems incomplete. Second, by the same token, the average rate of profit in equation (9) appears to be wrong, since it too should be defined in price of production terms. This is arguably more serious, because if the rate of profit is wrongly defined, its use to define prices of production will generate inconsistency. The transformation problem is therefore concerned with the following questions: under what circumstances, if any, should Marx's procedure be corrected? And if Marx's procedure is suitably corrected, under what circumstances, if any, can Marx's account of a capitalist economy in terms of a labor theory of value be maintained?

4. Classical Long-Period Equilibrium

Because many inputs have to be combined to produce output, if inputs are to be evaluated at prices of production, the input–output structure of the economy has to be specified in more detail. Let A_{ij} and L_j denote, respectively, the amounts of physical input i and labor used in the production of the total amount of good j. The corresponding amounts per unit of output are denoted as $a_{ij} = A_{ij}/Q_j$ and $l_j = L_j/Q_j$, and we shall denote the $n \times n$ input–output matrix and the $1 \times n$ vector of labor inputs, respectively, as $\mathbf{A} = [a_{ij}]$ and $\mathbf{l} = (l_1, \ldots, l_n)$. The ith column of $\mathbf{A}$ is denoted by $\mathbf{A}_{.i}$.[11] The vector of aggregate net output is $\mathbf{y} = (\mathbf{I} - \mathbf{A})\mathbf{Q}$, where $\mathbf{I}$ is the $n \times n$ identity matrix.[12]

Correcting Marx for incompleteness and inconsistency involves correcting equations (9) and (10). Let $\mathbf{p}^e$, w^e, and r^e denote the vector of production prices, and the corresponding (uniform) wage rate and profit rate, which are in principle distinguished from the market prices of the same variables, and may be interpreted as equilibrium values.

Axiom 1 (Long Period). *Long-period equilibrium prices ("prices of production" or "production prices") are the prices that support an equalized rate of profit:*

$$\mathbf{p}^e = (1 + r^e)(\mathbf{p}^e\mathbf{A} + w^e\mathbf{l}) \tag{11}$$

Equation (11) is a system of n equations in $n + 2$ unknowns. The system can be closed by specifying the value of either of the distributive variables, (r^e, w^e), and by choosing a numéraire, which determine equilibrium prices and the other distributive variable.[13] In this paper, we follow the literature and suppose that equation (11) is solved by specifying the wage rate. This can be done in a number of ways. One possibility is to take the money wage as given, which we explore in Section 6, so that it is the same whether prices are simple prices or prices of production. This implies that across the transformation, the real wage rate will change. Another possibility, which we consider in Section 5, is to specify the wage as a real wage, in terms of the commodities $\mathbf{b}$ the wage purchases, whether at simple prices or prices of production. This implies that across the transformation, the money wage rate will change.

However the wage is specified, aggregate equilibrium profits are total revenues less total costs, denoted as $\Pi^e = \mathbf{p}^e\mathbf{Q} - (\mathbf{p}^e\mathbf{AQ} + w^e L)$, and the rate of profit can be written as

$$r^e = \frac{\Pi^e}{\mathbf{p}^e\mathbf{AQ} + w^e L} \tag{12}$$

For most of the approaches we consider (but not all, as we shall see), there is no disagreement over Axiom 1. The issue that separates different approaches is rather how to specify a value theory that is compatible with Axiom 1.

5. The Transformation Problem as an Impossibility Result: The Dualist Approach

Equation (2) is explicit about the value of money, labor values, and prices, and thereby avoids dimensional confusion between units of money and units of labor time. However, if equation (2) holds for all i, then for any i and j, it must be the case that

$$\frac{p_i}{p_j} = \frac{\lambda_i}{\lambda_j} \tag{13}$$

Of course, letting commodity j be the money-commodity restores equation (2), and further assuming the value of money to be unity normalizes all labor values. But the dominant approach in the literature up to the 1970s treats this as a secondary consideration of choice of numéraire, and instead sees the statement of the labor theory of value directly as equation (13). The labor theory of value was taken as specifying that relative prices are determined by relative labor values. Notably, money is absent from this interpretation of Marx (notwithstanding Marx's own emphasis on money in the first three chapters of Marx, 1976). With money reduced to an arbitrary choice of normalization, there was no conceptual linkage between labor values and monetary prices. Instead, there was an underlying (intrinsic, invisible, and essential) system of labor values and associated exploitation, and a phenomenal (extrinsic, visible, and superficial) system of prices and profit rate. The question then was how the visible system could be derived from the invisible system. If Marx's method was wrong, both incomplete and inconsistent, then how should/could prices be derived from values? For around 80 years from the mid-1890s, the issue was posed and analyzed in this manner. With separate price and value systems, this tradition has become known as the "dualist" approach.

Since the value of a commodity is the sum of the value embodied in the means of production and the labor that works with those means of production, we can write the value equations as $\boldsymbol{\lambda} = \boldsymbol{\lambda}\mathbf{A} + \mathbf{l}$, so that labor values are uniquely determined by $\boldsymbol{\lambda} = \mathbf{l}(\mathbf{I} - \mathbf{A})^{-1}$. Hence, immediately, $\lambda\mathbf{y} = L = \mathbf{lQ}$, or the value of net output is the total hours worked. The value of the labor embodied in the means of production is just the value of constant capital, so that $c_i = \boldsymbol{\lambda}\mathbf{A}_{.i}$; and, as before, $v_i + s_i = l_i$.

Within the dualist tradition, a common approach has been to assume that workers do not save but spend all of their income, and to follow Marx (1976, chapter 6, pp. 274–275) in specifying the value of labor power as the value of the commodities purchased with the wage. Then the value of labor power (per hour of labor hired) is the value of the real wage (per hour), as in equation (5). Notice that this specification requires *equivalent exchange in all individual exchanges*, as in equation (2). It immediately follows that the rate of surplus value in equation (7) can be rewritten as

$$e = \frac{L - \boldsymbol{\lambda}\mathbf{b}}{\boldsymbol{\lambda}\mathbf{b}} = \frac{1 - (\boldsymbol{\lambda}\mathbf{b}/L)}{\boldsymbol{\lambda}\mathbf{b}/L} \tag{14}$$

which is nonnegative as long as $L - \boldsymbol{\lambda}\mathbf{b} \geqq 0$. Hence, the value equations could be written more fully as

$$\boldsymbol{\lambda} = \boldsymbol{\lambda}\mathbf{A} + (1+e)\boldsymbol{\lambda}\frac{\mathbf{b}}{L}\mathbf{l} \tag{15}$$

In this manner, the value equations are completely specified by the input–output structure of the economy, the labor coefficients, and the real wage. Hence, they are quite separate from anything to do with prices.

Axiom 2 (Dualism). *Value magnitudes are determined independently from price magnitudes. More specifically, for all i, (i) the value of constant capital is* $c_i = \boldsymbol{\lambda}\mathbf{A}_{.i}$*; (ii) the value of variable capital (value of labor power) is* $v_i = (\boldsymbol{\lambda}\mathbf{b}/L)l_i$*; and (iii) the total new value produced is equal to total direct labor employed,* $v_i + s_i = l_i$.

With the wage rate specified as whatever is necessary to purchase $\mathbf{b}/L$, the classical equation for prices of production, equation (11), becomes

$$\mathbf{p}^e = (1 + r^e)\left(\mathbf{p}^e\mathbf{A} + \mathbf{p}^e\frac{\mathbf{b}}{L}\mathbf{l}\right) = (1 + r^e)\mathbf{p}^e\mathbf{M} \tag{16}$$

where $\mathbf{M} = \mathbf{A} + \frac{\mathbf{b}}{L}\mathbf{l}$ is the augmented input coefficient matrix. The system of equations (16) is linear and homogeneous, and a necessary condition for a solution is that the determinant $|\mathbf{I} - (1 + r^e)\mathbf{M}|$ is zero. If $\mathbf{M}$ is indecomposable, then it has a unique positive eigenvalue $1/(1 + r^e)$ to which it can be associated a corresponding unique (up to a positive scalar) positive eigenvector $\mathbf{p}^e$.[14] Choosing a numéraire to close the system then completes the solution.

How then does this result bear on the derivation of the price system from the value system? The short answer is that it does not. Since the augmented input coefficient matrices for the value system $\mathbf{M} + e\frac{\mathbf{b}}{L}\mathbf{l}$ and for the price system $\mathbf{M}$ are different, then the solutions to equations (15) and (16) will be different, and so prices cannot be proportional to labor values. The point is that in the two systems, the nonwage net product is distributed differently. In the "value system," it is distributed in proportion to variable capital advanced (defined in Axiom 2), whereas in the "production price system," it is distributed in proportion to total capital advanced. The same physical quantity of nonwage net product (albeit differently evaluated in labor values and production prices) is distributed over different amounts of capital. So the price and value systems are different. This is summarized in the following theorem.

Theorem 1 (The Transformation Problem). Under **Dualism** and **Long Period**, it is generally impossible that relative prices are equal to relative values. In other words, the *strong equilibrium price view* is logically untenable.

Within the dual, long-period framework, a first reaction to Theorem 1 has been to abandon the *strong equilibrium price view* in favor of the *weak equilibrium price view* and maintain that although relative values are not necessarily equal to relative prices, they still determine them in some sense. While under Axioms 1 and 2, it has long been known that it is possible to derive a precise relation between prices of production and labor values (Pasinetti, 1977, Appendix to chapter 5; Roemer, 1981, section 8.2), this relation is far from the simple one that Marx proposed. For all sectors i, price-value differences depend on how the composition of capital in the production of i differs from that in the production of the commodity used as numéraire, both compositions being evaluated at prices of production. But they also depend upon the "the intricate network of relations between rate of profit and prices in the whole economic system" (Pasinetti, 1977, p. 136). Vector–matrix multiplication and matrix inversion might show a one-to-one correspondence between labor values and prices of production, but that correspondence is very much more complicated than the relatively simple relations which Marx adduced (see also Mohun, 2004).

Further, the weak equilibrium price view provides at best a partial answer to the problems raised by Theorem 1. For consider the aggregate proportionalities that for Marx anchored prices to labor values: whatever the complexities at the individual level, at the aggregate level, the relationship is simple. Let *TV* and *S* denote, respectively, the total amount of value and the aggregate surplus value produced in the economy. Similarly, let *TR* and Π denote, respectively, aggregate revenues and aggregate profits: these may be equilibrium amounts, or just observed market magnitudes.[15] Dimensionally, if price and value magnitudes are denominated, respectively, in money units and in labor time, then any relation between these value and price aggregates must be mediated by the value of money, and the aggregate relation between them can be stated as:

Axiom 3 (Aggregate Proportionalities). *Total revenue is proportional to total value and total profits are proportional to total surplus value. Formally, (i)* $\lambda_m TR = TV$ *and (ii)* $\lambda_m \Pi = S$.

Further, were it the case that either $\lambda_m = 1$, or labor values were defined in terms of money, or prices of production were defined in terms of labor hours, the dimensional distinction between monetary magnitudes and quantities denominated in labor time would be eliminated, and then we could write:

Axiom 4 (Aggregate Equalities). *Total revenue is equal to total value and total profits are equal to total surplus value. Formally, (i)* $TR = TV$ *and (ii)* $\Pi = S$.

While the issue of dimensionality is theoretically important, we do not discuss it at this point: we list both Axioms 3 and 4 and leave their variables undefined, allowing different approaches to adopt different notions of price and value magnitudes.

It may be objected that the relation between labor and monetary aggregates should not be considered as an axiomatic property. Rather, the existence of such a relation should be, and indeed usually is, proved as a *result* in a given economic environment, under certain conditions. Yet, its central relevance in value theory is such that "its epistemological status in our understanding is as a postulate. We seek a model which will make our postulated belief true" (Roemer, 1982, p. 152). For an axiomatic study involves the specification of a domain of problems, and the formulation of a list of desirable properties of solutions for the domain. Axioms 3 and 4 formalize one of the key properties of Marxian value theory.

This distinction between Axioms 3 and 4 makes little difference to our results here. For, in general, there is no scalar λ_m such that $\lambda_m \mathbf{p^e Q} = \boldsymbol{\lambda}\mathbf{Q}$ and $\lambda_m \Pi^e = S = \boldsymbol{\lambda}(\mathbf{Q} - \mathbf{AQ} - \mathbf{b})$: by Theorem 1, production prices are not proportional to labor values, and because there are n equations in equation (16) but $n+1$ variables, one further equation can be specified; this allows either (i) or (ii) in Axiom 3 but not both. Thus, a fortiori, Axiom 4 cannot hold either.

Specifying a numéraire amounts to choosing some commodity or composite commodity whose "value" is invariant to evaluation at simple prices and prices of production (which is why Seton, 1957, called the choice of a numéraire an "invariance postulate"). Since only one such numéraire can be chosen, this is clearly a serious embarrassment for the interpretation that prices of production are derived from labor values. For if the choice of numéraire is that total revenue is proportional to total value, then total profit will not be proportional to total surplus value, in which case the explanation of profit as originating in surplus value fails. Conversely, if the choice of numéraire maintains proportionality between total surplus value and total profits, then the macroeconomic labor theory of value fails.

The stronger version of the transformation problem can then be stated as follows.

Theorem 2 (The Strong Transformation Problem). Dualism, Long Period, and **Aggregate Proportionalities** are inconsistent. In other words, the *weak equilibrium price view* is logically untenable if either **Aggregate Proportionalities** or **Aggregate Equalities** is imposed.

Within the standard dualist approach to value theory, two main ways out of the impossibility highlighted by Theorems 1 and 2 have been suggested.[16] The standard solution has been to drop **Aggregate Proportionalities** (or **Aggregate Equalities**) and to deflate the relevance of the *weak equilibrium price view* to emphasize the *profit & exploitation view*. This is the literature on the so-called *Fundamental Marxian Theorem* (Okishio, 1963; Morishima, 1973, 1974; Roemer, 1981). For note that although Axiom 3 does not hold, comparison of the characteristic equations of the two systems shows immediately that, regardless of choice of numéraire: (i) $r^e > 0$ if and only if $e > 0$, (ii) $r^e < e$ (unless $\mathbf{A} = \mathbf{0}$), and (iii) each of r^e and e is a monotonically increasing function of the other.

The second solution originally proposed by Okishio (1963) and later developed by Morishima (1973, 1974) consists of a significant weakening of **Aggregate Proportionalities** (or **Aggregate Equalities**), requiring it to hold only at "the long-run equilibrium balanced-growth output vector (or the von Neumann equilibrium output vector)" (Morishima, 1974, p. 623). In fact, it is not difficult to show that if the aggregate output vector coincides with the column eigenvector associated with the largest positive eigenvalue of **M**, then total revenues equal total value and total surplus value equals total profits.[17] Furthermore, Morishima has shown that, under standard assumptions, there exists a dynamic process—an iteration procedure—whereby the economy reaches the state of long-run equilibrium balanced growth (or the von Neumann equilibrium), and the price and output vectors converge to the von Neumann equilibrium vectors.

Shaikh (1977) has also proposed an iterative approach, arguing that it resolves the incompleteness issue of Marx's own approach. He interprets the Marxian prices of production in equation (10) as the first stage of a recursive procedure that can be written as

$$\mathbf{p}^M_{k+1} = \left(1 + r^M_k\right) \mathbf{p}^M_k \mathbf{M}, \quad \text{for } k = 0, 1, \ldots \tag{17}$$

with $1 + r^M_k = \frac{\mathbf{p}^M_k \mathbf{Q}}{\mathbf{p}^M_k \mathbf{MQ}}$ for all $k = 0, 1, \ldots$ and initializing the iteration by setting $\mathbf{p}^M_0 = \lambda$. He then shows that in the limit, $\mathbf{p}^M_k$ and r^M_k converge to the dualist solutions $\mathbf{p}^e$ and r^e of equation (16). Further, equation (17) and the definition of r^M_k imply that $\mathbf{p}^M_{k+1}\mathbf{Q} = \mathbf{p}^M_k\mathbf{Q}$ is true at all stages k of the iteration and therefore part (i) of Axiom 3 (**Aggregate Proportionalities**) holds.

These iterative solutions do not really address the issues raised by the transformation problem because they do not provide a way out of the impossibility highlighted by Theorem 2. Morishima's solution based on the iterative procedure restricts the validity of value theory to a rather special case, namely, the von Neumann equilibrium, which is reached by a very specific dynamic process. While Shaikh's iterative approach is not restricted to a specific equilibrium path, this generality comes at a cost. First, while equation (17) shows how Marx's own "transformation procedure," once extended, can lead from simple prices to production prices, Theorem 2 remains valid and therefore part (ii) of Axiom 3 does not hold.[18] Second, as Shaikh (1977) acknowledges, it is well known that a characteristic equation such as equation (16) can be solved iteratively for $\mathbf{p}^e$ and r^e beginning from some arbitrary initial $\boldsymbol{p}_0$ and r_0. But then, it is quite unclear that his result identifies a specific, unique relation between values expressed as simple prices and prices of production.

More generally, both the fundamental Marxian theorem and the iterative solutions state how in general the value and price systems are related, but that is all. There are no causative relations between the two systems, which are different from each other and independent of each other. It remains true that all that is necessary to determine production prices is knowledge of the physical structure of the economy: the input–output coefficients, the labor coefficients, and the real wage. Value magnitudes play no role. Further, the same information that is needed to solve the value equations (15) is all that is needed to solve equation (16). The value equations are therefore redundant.

This was the position reached by the end of the 1970s. And yet, there are several oddities in this received view. For one example, money is a casual afterthought in the dualist approach, emerging out of a possible normalization. This seems to miss an empirically and theoretically essential aspect of capitalist economies. For another, the value of labor power is taken to be the value of the real wage (Axiom 2(ii)) and the same real wage is used to augment the input coefficient matrix. Hence, the real wage is held invariant to transformation, which implies that the money wage in the "price world" is different from the implicit money wage in the "value world." But no economic rationale is given for this latter. For reasons such as these, more recent approaches question the coherence of this dualistic separation of the "value world" from the "price world," and generically, they have come to be known as "*single-system*" approaches. Because they question this separation in different ways, they each have a different set of axioms from those of the dualist approach and from each other.

6. The New Interpretation

The *new interpretation* (henceforth, NI) has been proposed, independently, by Duménil (1980, 1984) and Foley (1982, 1986).[19] For them, the fundamental question is not how to derive prices of production from values that are prior in some sense, but rather how the theory of exploitation, based on the value difference between labor power and labor, is compatible with the theory of capitalist competition. But while the theory of exploitation is essential to the understanding of capitalism, the theory of competition embraces more than the characterization of a long-run price equilibrium. So, combining the theory of capitalist exploitation with the theory of capitalist competition must show the compatibility of class exploitation with each and every price system (of which a long-run equilibrium price system is but one example). Consequently, the NI adopts both a descriptive interpretation of value theory as providing a broad theoretical framework for "understanding the dynamics of accumulation and distribution in capitalist-commodity producing economies" (Foley, 2014, p. 17), and a *profit & exploitation view*.

The NI formulation proceeds in two steps. First is an interpretation of the labor theory of value. For the individual commodity, the labor theory of value is specified by equation (2) for some definition of the value of money. This is how it appeared in Smith's precapitalist "early and rude state," as a "commodity law of exchange;" Ricardo extended this to a capitalist economy with produced means of production and found that, in general, such a commodity law of exchange did not hold. It was rather superseded by the "capitalist law of exchange," specified as the determination of prices that supported an equalized rate of profit.[20] Marx then tried to show that the capitalist law of exchange merely modified the commodity law of exchange in the sense that it took value from where it was produced and redistributed it according to total capital advanced. In Marx's particular procedure, the deviations generated by this redistribution summed to zero, and the NI argues that, in a certain (ontological) sense, they could not meaningfully do anything else. It is this insight that motivates the NI, and it is specified accordingly as a fundamental *conservation principle*.

Axiom 5 (Conservation Principle). *For any specification of prices, the total value created by labor in all value-creating production processes is conserved in exchange. Formally,*

$$\mathbf{py} = \frac{\boldsymbol{\lambda}\mathbf{y}}{\lambda_m} = \frac{L}{\lambda_m} \tag{18}$$

Two features of Axiom 5 should be emphasized. First, it says that when the commodity law of exchange (equation (2)) does not in general apply for each individual commodity, it nevertheless *does* apply for the aggregate of commodities in net value added; their value in aggregate is conserved in exchange. Thus, at its core, the NI adopts a *macroeconomic view*.[21] Second, Axiom 5 *defines* the value of money in the NI:

$$\lambda_m = \frac{L}{\mathbf{py}} \tag{19}$$

The second step of the NI concerns the specification of the value of labor power. The reason that equation (2) does not in general apply to individual commodities is because of their different production conditions (their different compositions of capital). So, in general, individual commodities must exchange at prices different from their simple prices in order that value is redistributed through exchange. This must apply to each and every commodity purchased by the wage. Hence, if all the wage is spent,

$$\mathbf{p}\frac{\mathbf{b}}{L} \neq \frac{\boldsymbol{\lambda}}{\lambda_m}\frac{\mathbf{b}}{L} \tag{20}$$

Yet labor power itself is not a produced commodity, there is no composition of capital involved, no rate of profit in its production that competition will tend to equalize, and hence no price of production of labor power. Consequently, in the sale of labor power for a wage, the law of commodity exchange continues to apply. That is,

Axiom 6 (Weak Single-System). *In the sale of labor power for a wage, the capitalist law of exchange has no effect, and the commodity law of exchange continues to apply so that*

$$w = \frac{vlp}{\lambda_m} \tag{21}$$

In sum, because of different compositions of capital, equation (2) cannot in general hold. But for the NI, equation (2) does continue to hold, first for the aggregate of commodities in value added (Axiom 5); and second, in the sale of labor power for a wage (Axiom 6). These are the key features of the NI, summarized in the following theorem.

Theorem 3 (New Interpretation). Under **Conservation Principle** and **Weak Single-System**, the *profit & exploitation view* is logically consistent. Furthermore, aggregate net output is proportional to total value added and aggregate profit is proportional to aggregate surplus value.

Observe that, since the left-hand sides of equations (20) and (21) are equal if all the wage is spent, then Axiom 2(ii) in the dualist approach cannot hold. Thus, in specifying the value of labor power as the value of the real wage rate, the dualist approach presumes equal exchanges in what the wage is spent on, and *that is precisely what cannot be the case*. The NI argues that the dualist approach is therefore incoherent in its treatment of the wage rate and the value of labor power. Rather than the real wage, the NI takes the money wage rate as given. By equation (21), the given money wage rate determines the value of labor power, and, using equation (19), the value of labor power can be written as

$$vlp = \frac{wL}{\mathbf{py}} \tag{22}$$

As for prices of production, the NI formulation is the same as equation (11) in Axiom 1, but without specifying w^e in terms of the workers' consumption bundle. There is no requirement in the NI either to specify the wage as what it is spent on, or indeed to presume that all of the wage is spent. Then, from equation (11), one can derive a one-to-one inverse relation between the wage rate and the profit rate that, by equation (22), can be specified in terms of the value of labor power. The higher the value of labor power, the lower the rate of profit. But once the value of labor power is fixed, then the rate of profit and the corresponding production prices can be derived.

In one sense, there is little substantive difference between the dualist prices of production and the NI prices of production in that both derive from equation (11) in Axiom 1, for appropriate specifications of the wage rate. But consider again equation (22). This says that the value of labor power is the wage share of value added *at any set of prices*. This result is central to the NI because it demonstrates the existence of class exploitation. Using Marx's metaphor, the value of labor power divides the total "working day" into a period of "necessary labor" in which the working class produces a value equivalent to its wages, and a further period of "surplus labor" in which it produces a value that is expropriated by the capitalist class as profit. At any moment in time, class struggle (for example, over the social norms for the reproduction of the working class) determines how this "working day" is divided. That is, class struggle determines how much of the net value added the working class produces is won back in the form of wages, and this class struggle determination of the wage share is no different whether prices are simple prices or production prices. As long as exploitation exists, the wage share must be less than one.

The NI thereby shows that *the existence of capitalist exploitation is independent of any account of price formation*. That is, in the NI, equation (22) always holds, whereas equation (11) might not. Hence, the NI is also a framework for empirical analysis, since the total number of hours worked, aggregate value added in price terms, and the average hourly wage rate are all measurable quantities.

One may object that, albeit formally correct, the previous argument does not clarify why one needs the labor theory of value to capture exploitative relations. For, in the NI, the existence of exploitation at the aggregate level reduces to the fact that the wage share is less than 1. This objection is not entirely convincing. The NI does not *define* exploitation as corresponding to a wage share smaller than 1: the equivalence, at the aggregate level, between exploitation and the existence of profits is *derived* from more primitive axioms concerning the nature and determinants of value and surplus value. Further, such equivalence does not imply that monetary phenomena are all that matters while the notion of exploitation, and labor accounts, are irrelevant. In recent work, for example, Veneziani and Yoshihara (2015, 2017) and Yoshihara (2010) have extended the NI in order to define exploitation at the level of individual agents and social classes. They have shown that even at the micro level, the existence of exploitation is synonymous with positive profits. The exploitation status of individuals and classes, and labor accounts more generally, provide important positive and normative insights on capitalist economies and their class structure, which are not reducible to the wage share being smaller than 1. Further, they have shown in Yoshihara and Veneziani (2013) that, contrary to a common view, the concept of exploitation is not just a complicated way of capturing the productivity of the economy. Under the NI, the existence of profits is *not* synonymous with the existence of a surplus denominated in any arbitrary commodity (as the *Commodity Exploitation Theorem* implies, for example, in Roemer, 1981). Rather, a wage share smaller than 1 is synonymous with the exploitation of *labor*.

As a theory, however, the NI is incomplete for two reasons. First, there is no theoretical determination of *vlp* other than in the general terms of class power and class struggle. And second, while it defines λ_m as equation (19), it has no theoretical account either of its formation or its movement over time. While equation (19) does imply that λ_m will fall through time because of both productivity increases and pure price inflation, there is no account of pure price inflation. While this incompleteness detracts neither from the generality of the NI as an account of exploitation, nor from its usefulness as a foundation for empirical analysis, it nonetheless requires further theoretical development. As such, it specifies a progressive research agenda.

7. An Althusserian Approach

A rather different interpretation to the NI has been proposed by Wolff *et al.* (1982, 1984a, 1984b) (henceforth, WCR) and further extended and generalized by Roberts (1997, 2009). WCR see themselves as "applying the perspectives and insights of the Althusserian tradition to the reinterpretation of Marx's theoretical and economic texts" (Wolff *et al.*, 1982, p. 565). Within this framework, the notion of causation implicit in the dualist approach is rejected as both reductive and essentialist, resting on some essence determining some consequent (such as in the standard reading of Marx, values determining prices). In its place is a focus on "overdetermination": mutual and reciprocal determination together with relations of constitutivity. Constitutivity is "the power of each aspect of society not merely to affect other aspects, but also to effect them, constitute them, participate in determining the nature of, as well as the changes in, every other aspect" (Wolff *et al.*, 1982, p.565). Because production and circulation are both overdetermined, the concepts of value and price, understood as the form that value takes in exchange, are interdependent, and constitute each other. They further change according to the degree of complexity of the economic processes that actualize class relations, so that discourses themselves are changed.

*I*n particular, the concepts of *Capital III* are, taken together, a different discourse from those of *Capital* I; the new determinations of *Capital III* (such as interindustry competition) require new concepts (such as the average rate of profit), and the changing discourse requires corresponding changes in the meanings of value and the form it takes in exchange. Because *Capital I* constructs capitalist class relations to show how surplus value derives from unpaid labor time, and because *Capital III* shows how the form of surplus value, as profit, is also a relation between paid and unpaid labor time, then WCR are adopting a *profit & exploitation* view.[22]

Within this methodological approach, value is "the quantity of social labor-time 'attached to' the commodity in production, given the nature and functioning of the processes involved in commodity circulation. The form of value in exchange is ... the quantity of social labor-time 'attached to' the commodity in circulation, given the particular processes of production" (Wolff *et al.*, 1984a, p. 123). Value and value-form are equal in *Capital I*, but only as a preliminary step. In general, and in actual capitalist economies, they differ quantitatively, both being jointly determined by production and circulation conditions. It follows that, while "value-form" is a price, it is a price denominated in labor time rather than in money. Given their focus on individual values and prices, WCR adopt a *microeconomic view*. More precisely, WCR adopt Axiom 1 (**Long Period**), but interpret it differently. For them, prices of production are the magnitudes of *labor time* that allow the reproduction of the capitals of each industry with a uniform profit rate. Hence, they can be called "labor prices of production" and denoted $\mathbf{p}^{wcr}$. Thus, in the discourse of *Capital III*, Axiom 1 becomes the basic statement of the value-form, with its production prices $\mathbf{p}^{wcr}$ measured in labor time.[23] Interpreted in this way,[24] Axiom 1 both constitutes and is constituted by the basic statement of value that is written as Axiom 7:

Axiom 7 (Strong Single-System). *The value of each commodity is the sum of the prices of production of its constant capital plus the living labor required. Formally,*

$$\boldsymbol{\lambda} = \mathbf{p}^{wcr}\mathbf{A} + \mathbf{l} \tag{23}$$

Letting $\mathbf{p}^{e} = \mathbf{p}^{wcr}$, equations (11) and (23) provide a codetermination of value and value-form. Together they define a system of $2n$ equations in $2n + 1$ unknowns ($\boldsymbol{\lambda}$, $\mathbf{p}^{wcr}$, and r), which can be solved in the same manner as dualism's equation (16). Then a unique normalization can be specified by defining the rate of profit as a ratio of labor amounts as follows.

Axiom 8 (Labor Prices of Production). *The rate of profit is the ratio of total unpaid labor to total capital advanced in labor time. Formally,*

$$r = \frac{L - \mathbf{p}^{wcr}\mathbf{b}}{\mathbf{p}^{wcr}\mathbf{AQ} + \mathbf{p}^{wcr}\mathbf{b}} \tag{24}$$

Letting $\mathbf{p}^{e} = \mathbf{p}^{wcr}$, by equations (11) and (24), it follows that

$$\mathbf{p}^{wcr}\mathbf{Q} = \mathbf{p}^{wcr}\mathbf{AQ} + L \tag{25}$$

which implies that

$$\mathbf{p}^{wcr}\mathbf{y} = L \tag{26}$$

Equation (26) expresses "a necessary equality between ... the direct labor-time expression of the net product ... and ... the expression in labor-time terms for the revenues which are realized by the two classes together when that net product is distributed between them through the circulation process" (Wolff *et al.*, 1982, p. 579).[25]

On the basis of the foregoing, and noting that $\mathbf{p}^{wcr}\mathbf{Q} = \boldsymbol{\lambda}\mathbf{Q}$ follows immediately by postmultiplying equation (23) by $\mathbf{Q}$ and comparing with equation (25), the WCR Theorem can be stated as follows.

Theorem 4 (Single System). Let $\mathbf{p}^{e} = \mathbf{p}^{wcr}$. Under **Long Period, Strong Single-System**, and **Labor Prices of Production**, the *profit & exploitation view* is logically consistent. Furthermore, **Aggregate Equalities** is satisfied.

WCR conclude that the traditional interpretation that "a valid Marxian transformation must explain prices and the rate of profit as exclusively determined by physically embodied direct and indirect labor time ... is ... *not* the only basis on which to confront the price-value relation. Reading Marx's *Capital* as expressing a view of the role of labor-time categories which is quite thoroughly opposed to the Ricardian approach in all its variants has allowed us to resolve the traditional puzzles of the transformation problem by posing them in different fashion" (Wolff *et al.*, 1984b, pp. 435–436).

However, two features of the WCR system should be emphasized that raise doubts concerning this conclusion and more generally, the WCR approach to Marxian value theory. First, the role of equation (23) in the WCR system is unclear, because it simply adds n more variables and n more equations. It is therefore not completely obvious that equations (11) and (23) adequately fulfill the constitutive roles that WCR allocate for them. Conceptually, equation (23) can be interpreted as a part of a complete value accounting system in which constant capital, variable capital, and surplus value are all expressed in value magnitudes. Yet, formally, values are defined purely *ex post* and play no role either in the definition or in the determination of any other variable in the WCR system. WCR solve equations (11) and (24) for $\mathbf{p}^e = \mathbf{p}^{wcr}$ and r, and the value equations (23) are irrelevant to that solution.[26] Notice further that, directly from equation (24), total profit in labor units and total surplus value are the same. So the only relevance of the value equations (23) is in showing that Axiom 4(i) holds. If that is all that equation (23) is good for, its status as a fundamental constitutive relation seems somewhat artificial.

Second, money plays no role in Axioms 1 and 7–8. While prices of production are denominated in labor times, one of these prices (say, the kth) will be a labor-time price of gold; dividing all other labor-time prices by this labor-time price of gold translates labor-time prices into money prices, so that $P_j = \frac{p_j^{wcr}}{p_k^{wcr}}$, which is the WCR interpretation of Marx's equation (2) in an overdetermined *Capital III* world. Yet, as in the dualist interpretation—and unlike in other approaches—money plays no direct role in the WCR system, for its analysis of labor time accounting is independent of and prior to the introduction of money and the definition of its value.

8. A Macromonetary Approach

Moseley (2016) also offers an explicit methodological account of values and prices. He proposes that Marx's analysis be interpreted as focusing first on the production of surplus value, and then on its distribution, so that a sequential (rather than a simultaneous) account is necessary. This contrasts, for example, with WCR, who see both value and its form (price) as being simultaneously determined for a given level of abstraction. Moseley offers an interpretation that sees first a macrodetermination of total surplus value for the economy as a whole, and then a microdetermination of how this total is divided between different industries. Given the logical primacy of the macrorelations, we characterize Moseley as adopting a *macroeconomic view*.

8.1 *The Macrodetermination of Surplus Value*

Moseley stresses an interpretation of the circuit of capital that sees a given amount of money advanced as (constant and variable) capital that reproduces itself together with an increment called surplus value. Because it is this latter that has to be explained, the money advanced is taken as given. His approach implies that neither equation (2) nor its implication, equation (13), represents Marx's labor theory of value. Instead, the latter is solely concerned with the determination of aggregate surplus value in money terms on the basis of aggregate money capital advanced. While "aggregate" might be taken to imply that something is aggregated, Moseley denies this on methodological grounds. Obviously, the money advanced is spent on definite quantities of inputs, but what is purchased is a microeconomic issue that cannot be considered until aggregate surplus value is first determined. So while inputs are purchased at unit prices that are presumed to be prices of production, these latter are (methodologically) posited yet undetermined and so cannot be explicitly considered at this macroeconomic stage. Similarly, since these prices determine the quantities that are purchased, those quantities cannot be explicitly considered. All that can be considered is the total quantity of money laid out, and its division into what is spent on means of production, and what is spent on labor power.

Let $C^{\$}$ and $V^{\$}$ denote the total amounts of money advanced as constant and variable capital, respectively, and let $S^{\$}$ denote the money surplus value produced. By definition, money surplus value is just the difference between total revenue $M^{\$}$ and the capital advanced to produce it:

$$S^{\$} = M^{\$} - C^{\$} - V^{\$} \tag{27}$$

The new value created in the circuit of capital is L, the sum of necessary labor V^{hrs}, and surplus labor S^{hrs}. As in the NI, Moseley then imposes Axiom 5, the **Conservation Principle** ("the key assumption in Marx's labor theory of value" [Moseley, 2016, p. 31]), according to which aggregate new value produced is proportional to total labor L. In the notation of this section,

$$V^{\$} + S^{\$} = \frac{(V^{hrs} + S^{hrs})}{\lambda_m}$$

Consequently,

$$S^{\$} = \left(\frac{V^{hrs}}{\lambda_m} - V^{\$} \right) + \frac{S^{hrs}}{\lambda_m}$$

Moseley then applies Marx's definition of necessary labor as a *macroeconomic* definition.

Axiom 9 (Necessary Labor). *Necessary labor in the aggregate is the labor that produces the monetary equivalent of the total capital advanced as wages, so that*

$$V^{hrs} = \lambda_m V^{\$} \tag{28}$$

Combining Axiom 9 with Axiom 5 immediately yields Moseley's main result:

$$S^{\$} = \frac{S^{hrs}}{\lambda_m} \tag{29}$$

Theorem 5 summarizes these results.

Theorem 5. Under the **Conservation Principle** and **Necessary Labor**, the *profit & exploitation view* is logically consistent. Furthermore, aggregate net output is proportional to total value added and aggregate profit is proportional to aggregate surplus value.

For Moseley, equation (29) is "Marx's 'surplus labour' theory of surplus-value ... the main conclusion of Volume I" (Moseley, 2016, p. 34) and hence the basic statement of Marx's labor theory of value.

Theorems 5 and 3 are remarkably similar. They both rely on the **Conservation Principle** (Axiom 5). Furthermore, the NI's Axiom 6 and Moseley's Axiom 9 can hardly be considered as radically different. Formally, Axiom 6 appears to be stronger than Axiom 9: whereas the NI applies equation (2) to the commodity labor power in Axiom 6, the NI's use of an hour as the basic unit is arbitrary, and multiplying equation (21) through by the total number of hours hired yields equation (28). The converse is also true however. Moseley's methodological emphasis that the various monetary aggregates are scalar magnitudes because prices are as yet unspecified is irrelevant in the treatment of $V^{\$}$. For with respect to variable capital in money terms, there are no undetermined prices. Moseley begins with the "per-worker-day" relation between variable capital as wages and variable capital as necessary labor, aggregates up across all "worker-days," and then treats the resultant aggregate relation as a given. But since the wage rate is known, one can also proceed in the opposite direction and so Moseley's Axiom 9 implies the NI's Axiom 6. Thus far then, the analytical difference between Moseley's interpretation and the NI is slight.

8.2 *The Distribution of Surplus Value*

Having determined total surplus value in money terms, $S^{\$}$, by equation (29), the general rate of profit r^{mos} is then defined as the ratio of $S^{\$}$ to the total money capital advanced:

$$r^{mos} = \frac{S^{\$}}{C^{\$} + V^{\$}} \tag{30}$$

This, in turn, is used to define Moseley's prices of production:

$$P_i^{mos} = M_i^{\$} = \left(C_i^{\$} + V_i^{\$}\right)(1 + r^{mos}) \tag{31}$$

Notice that first, r^{mos} is determined prior to prices of production; second, the inputs for each industry are commodities purchased at already existing prices (of production), so that there is nothing to transform; and third, these prices of production are not unit prices but industry gross revenues. These latter are the money capital advanced in industry i plus a portion of total money value produced, that portion being determined by the ratio of money capital advanced in industry i to that advanced in the economy as a whole (because equation (31) can be written as $P_i^{mos} = M_i^{\$} = \frac{(C_i^{\$}+V_i^{\$})}{(C^{\$}+V^{\$})}(C^{\$} + V^{\$} + S^{\$})$).

Moseley's sequential approach here appears to diverge from the NI (compare equation (11) for a given money wage and equation (31)). His rate of profit is formed out of the aggregate money magnitudes of *Capital I*, and, for Moseley, these latter are aggregates of quantities priced at prices of production. Hence, equation (31) does not determine prices of production; it only serves to distribute aggregate money surplus value among the various individual capitals. Yet, converting equation (31) into unit levels, it is easy to see that it can only do that if equation (30) is, in fact, that equalized rate of profit that forms prices of production at unit level. For prices of production have been posited at the outset, and the mathematics only allows for one solution value of the rate of profit.

Moseley further argues that Marx's two aggregate proportionalities are always both satisfied, because in his framework, they are not equations but identities. By Theorem 5, part (ii) of Axiom 3 (**Aggregate Proportionalities**) is satisfied. And the total revenues of *Capital I* are identical to the total revenues of *Capital III* because everything is denominated (whether implicitly or explicitly) in prices of production. But that is neither part (i) of Axiom 3 nor part (i) of Axiom 4 (**Aggregate Equalities**), both of which concern the relation between total revenue at *Capital III* prices and total value measured in hours.

Since Moseley proposes that his interpretation is what Marx himself wrote/meant, he has to specify some relation between *Capital I* (or, identically, *Capital III*) total revenues and total value measured in hours. There is obviously no difficulty in showing this for value-added, for that is what equations (28) and (29) do. But no deduction is possible for total revenue and total value, since $C^{\$}$ is denominated in prices of production that are not proportional to labor values. Hence, he requires a further axiom.

Axiom 10 (Constant Capital Proportionality). *The labor value of constant capital is imputed as*

$$C^{mos,hrs} = \lambda_m C^{\$} \tag{32}$$

Then, the combination of equations (18), (28), and (32) shows that part (i) of Axiom 3 is also satisfied, yielding the following theorem.

Theorem 6 (Moseley). Under the **Conservation Principle, Necessary Labor**, and **Constant Capital Proportionality**, the *profit & exploitation view* is logically consistent. Furthermore, **Aggregate Proportionalities** holds.

Theorem 6 is distinctive of Moseley's approach, and it shows that in his framework, both of Marx's aggregate proportionalities hold. Nonetheless, compared with Theorem 5, this comes at a significant cost, since it requires an arbitrary redefinition of the value of constant capital. While Axiom 10 provides a symmetry to Moseley's treatment of constant capital, variable capital, and surplus value, the interpretation of $C^{mos,hrs}$ is unclear, for this imputed value of constant capital is *neither* the hours historically necessary *nor* the hours currently necessary to produce the means of production (Moseley, 2016, pp. 259–260). It is rather a quantity of hours that is wholly determined by the prevailing value of money and the aggregate amount of money advanced as constant capital. Its interpretation is therefore obscure, and the argument appears contrived, suggesting that the only reason behind the redefinition of constant capital in Axiom 10 is to achieve a claim of Marxist textual fidelity via Theorem 6. Setting aside the theoretically questionable Axiom 10 and focusing on Theorem 5, Moseley's approach is not substantially different from the NI.

9. The Temporal Single-System Interpretation

An approach to Marxian value theory that has recently attracted both attention and controversy is the Temporal Single-System Interpretation (TSSI) (Freeman, 1996; Kliman and McGlone, 1988, 1999; Kliman, 2001, 2007). The TSSI adopts a *profit & exploitation view* and a *microeconomic view*, and supports a *weak price view*, arguing that values determine prices, although these are not equilibrium prices.

Consider the production period t. As production takes time, one can distinguish between the beginning of t, when inputs are bought, and the end of t/beginning of $t+1$ when outputs emerge from production and are sold. So far, we have not made this distinction because all of the main approaches (and indeed almost all schools of economics) evaluate inputs at current or replacement cost rather than historical cost. One reason is that we are interested in firms as going concerns, and in a situation in which prices are changing, we want to know whether the firm is viable and can reproduce itself. With a labor theory of value, there is another reason: we want to be able to attribute the value of net output to the labor that produced it. The TSSI insists on a temporalism and historical cost pricing (Kliman and McGlone, 1999, p. 34) and we write the temporalist perspective as the following axiom.

Axiom 11 (Temporalism). *In every production period t, the values and prices of inputs are determined at the beginning of t, before the values and prices of outputs, which are determined at the end of t/beginning of $t+1$, so that the former are determinants of the latter.*

Axiom 11 is supposed to capture the inherently dynamic nature of capitalist economies. In the rest of this section, the time subscript t refers to the beginning of production period t, whereas we use the time subscript $t+1$ to denote the end of period t and beginning of $t+1$.

The TSSI further rejects the view that labor values and monetary prices emerge from separate systems. This is instantiated in two different axioms. The first one states that money magnitudes enter the determination of values.

Axiom 12 (Value Single-System). *Price magnitudes enter the definition of values. More specifically, for all i and t, (i) the value of constant capital is $c_{it} = \lambda_{m,t}\mathbf{p}_t\mathbf{A}_{.it}$; and (ii) the value of variable capital is $v_{it} = \lambda_{m,t}w_t l_{it}$.*

Assuming that total new value produced in every sector is equal to total direct labor employed, $v_{it} + s_{it} = l_{it}$, Axiom 12 immediately implies that equation (6) becomes

$$\boldsymbol{\lambda}_{t+1} = \lambda_{m,t}\mathbf{p}_t\mathbf{A}_t + \mathbf{l}_t \tag{33}$$

The second part of the TSSI rejection of dualism is a stronger claim on the relation between prices and values.

Axiom 13 (Price Single-System). *Values and prices differ because of random, sector-specific deviations. Formally, for all t, there exists a vector $\mathbf{g}_t = \lambda_{m,t+1}\mathbf{p}_{t+1} - \boldsymbol{\lambda}_{t+1}$ such that $\mathbf{g}_t\mathbf{Q}_{t+1} = 0$.*

Combining Axioms 12 and 13, it immediately follows that

$$\lambda_{m,t+1}\mathbf{p}_{t+1} = \lambda_{m,t}\mathbf{p}_t\mathbf{A}_t + \mathbf{l}_t + \mathbf{g}_t \tag{34}$$

Thus, assuming workers to spend all their income, so that $w_t = \mathbf{p}_t\frac{\mathbf{b}_t}{L_t}$, aggregate nominal profits are $\Pi^N_{t+1} = (\mathbf{p}_{t+1} - \mathbf{p}_t\mathbf{A}_t - \mathbf{p}_t\frac{\mathbf{b}_t}{L_t}\mathbf{l}_t)\mathbf{Q}_{t+1}$. Let $i_{t+1} = \frac{\lambda_{m,t}}{\lambda_{m,t+1}} - 1$, where i_{t+1} is the TSSI inflation rate. Then, aggregate real profits, Π^R_t, and aggregate surplus value, $S_{t+1} = \mathbf{s}_t\mathbf{Q}_{t+1}$, are defined as follows:

$$\Pi^R_{t+1} = \left[\frac{\mathbf{p}_{t+1}}{1+i_{t+1}} - \mathbf{p}_t\mathbf{A}_t - \mathbf{p}_t\frac{\mathbf{b}_t}{L_t}\mathbf{l}_t\right]\mathbf{Q}_{t+1} \tag{35}$$

$$S_{t+1} = \mathbf{l}_t\mathbf{Q}_{t+1} - \lambda_{m,t}\mathbf{p}_t\mathbf{b}_t \tag{36}$$

That profits are defined in real terms is important in that the aggregate proportionality of profits and surplus value is interpreted in terms of real profits, not nominal profits (see Theorem 7 below).

Axioms 12 and 13 concern the relation between labor values and *market* prices. As concerns prices of production and the general profit rate, the TSSI makes two assumptions. First, it has a rather specific view concerning the determination of the general profit rate, which can be formally put as follows.

Axiom 14 (TSSI Profit Rate). *The general profit rate is given by the ratio between aggregate surplus value (converted into monetary units) and the historic, market cost of advanced inputs. Formally,*

$$r^{TSSI}_t = \frac{\mathbf{s}_t\mathbf{Q}_{t+1}}{\lambda_{m,t}\left(\mathbf{p}_t\mathbf{A}_t + \mathbf{p}_t\frac{\mathbf{b}_t}{L_t}\mathbf{l}_t\right)\mathbf{Q}_{t+1}}, \quad \text{for all } t. \tag{37}$$

The second assumption specifies prices of production not as those supporting a long-period equilibrium but as determined by a markup on historic market prices.

Axiom 15 (TSSI Production Prices). *Production prices are derived from applying the average profit rate to historic costs evaluated at past market prices. Formally,*

$$\mathbf{p}^{TSSI}_{t+1} = \left(1 + r^{TSSI}_t\right)\mathbf{p}_t\left(\mathbf{A}_t + \frac{\mathbf{b}_t}{L_t}\mathbf{l}_t\right), \quad \text{for all } t. \tag{38}$$

Based on this axiomatic system, TSSI proponents maintain that the literal truth of *all* of Marx's propositions can be shown:

Theorem 7 (TSSI). Assume that $\lambda_{m,t} > 0$ all t. Then, under **Temporalism, Value Single-System, Price Single-System, TSSI profit rate**, and **TSSI production prices**, "(a) all of Marx's aggregate value-price equalities hold; (b) values cannot be negative; (c) profit cannot be positive unless surplus value is positive; (d) value production is no longer irrelevant to price and profit determination; (e) the profit rate is invariant to the distribution of profit; (f) productivity in luxury industries affects the general rate of profit" (Kliman and McGlone, 1999, p. 55).

Claims (a) and (c) follow immediately from Axiom 13: postmultiplying equations (33) and (34) by $\mathbf{Q}_{t+1}$, using equations (35) and (36), and noting that $\mathbf{g}_t\mathbf{Q}_{t+1} = 0$ by assumption, it follows that $\lambda_{m,t+1}\mathbf{p}_{t+1}\mathbf{Q}_{t+1} = \boldsymbol{\lambda}_{t+1}\mathbf{Q}_{t+1}$ and $\lambda_{m,t}\Pi^R_t = S_t$, for all t. Claim (b) follows from equation (33) by assuming $\mathbf{p}_t$ to be nonnegative at all t. Claims (e) and (f), and the part of claim (d) concerning the profit rate, immediately follow from Axiom 14.[27] It is also easy to show that in the TSSI, the key claims of price and value theory also hold if production prices are considered instead of market prices.[28]

In all frameworks considered so far, prices of production are the long-run prices that support an equalized rate of profit, and consequently they are equilibrium prices. But the TSSI axioms do not specify what is to be regarded as equilibrium in its temporal framework.[29] Indeed, the vector $\mathbf{p}_{t+1}^{TSSI}$ is determined on the basis of a uniform profit rate, a long-run condition which the TSSI regards as "a very particular case" (Kliman, 2001, p. 99), or a rather restrictive postulate (Freeman, 1996); yet this holds in the TSSI even outside a steady state by assuming that the profit rate is an average rate of profit. But "If market prices do not coincide with prices of production, there is no reason to think that the profit rate will be uniform across sectors. To assume a uniform profit rate in such circumstances amounts to imposing an arbitrary condition on the sectoral mark-ups" (Mongiovi, 2002, p. 408).

Kliman and McGlone deny that the TSSI "eliminates the inconsistency in Marx's value theory by supplying extra unknowns, in effect by modeling a perpetual disequilibrium in which 'anything goes'" (Kliman and McGlone, 1999, p. 50), because $\mathbf{p}_t$ and r_t^{TSSI} are determined prior to $\mathbf{p}_{t+1}^{TSSI}$, and thus in equation (38), there are n equations and n unknowns. Yet, despite the large number of assumptions, the formal structure of the TSSI is underdetermined. Consider the relation between TSSI values and market prices. At a steady state, equations (35) and (36) become

$$\boldsymbol{\lambda} = \lambda_m \mathbf{pA} + \mathbf{l} \tag{39}$$

$$\lambda_m \mathbf{p} = \lambda_m \mathbf{pA} + \mathbf{l} + \mathbf{g} \tag{40}$$

But then there are $n + 1$ degrees of freedom, unless *first*, it is assumed that in a steady state $\mathbf{g} = \mathbf{0}$, or equivalently, that $\boldsymbol{\lambda} = \lambda_m \mathbf{p}$, so that goods exchange at simple prices, and *second*, a formal definition of λ_m is provided. As regards the first point, since $\mathbf{g}_t$ is determined after market prices are realized, the alternative to value-price proportionality is to deny the steady state so that prices determine values "historically." But then, all variables are determined *ex post* by observed, unexplained market prices, with little explanatory power. As regards the second point, there is no definition of the value of money in the TSSI and so the model is undetermined. To assume $\lambda_{m,t} = 1$, all t, and state that this implies no loss of generality (e.g., Kliman and McGlone, 1999, p. 36) is unconvincing. In equations (39) and (40), *if* one assumes $\mathbf{g} = \mathbf{0}$ to avoid underdetermination, then the choice of λ_m is largely immaterial in that commodities are already assumed to exchange at their simple prices. But outside of a steady state, it is difficult to justify the assumption that $\lambda_{m,t} = 1$, all t.[30] The absence of a definition of $\lambda_{m,t}$ casts some doubt on Theorem 7, which crucially rests on the assumption that the undefined variable $\lambda_{m,t}$ is positive at all t.

That an explicit definition of $\lambda_{m,t}$ is unnecessary to prove Theorem 7 highlights some conceptual differences with competing approaches. In the NI, for example, prices and values are distinct and $\lambda_{m,t}$ is used "to move back and forth between money and labour accounts" (Foley, 2000, p. 7). Moreover, it is Axioms 5 and 6, and the specific definition of $\lambda_{m,t}$ that follows from them, that make it possible in the NI to retain "the central ideas of the labor theory of value, ... [although] they cannot and do not retain all of the results that hold when prices are proportional to labor values" (Foley, 1982, p. 42). In the TSSI, instead, there exist no distinct money and value accounts, and the single-system qualification reduces to the assumption that, apart from out-of-steady-state deviations, values are proportional to market prices. Thus, as shown by equations (39) and (40), $\lambda_{m,t}$ is just an undefined factor of proportionality between values and prices, which can be arbitrarily (and, from the TSSI standpoint, without loss of generality) assumed equal to unity.

Temporalism, "disequilibrium," and the extra unknowns, $\mathbf{g}_t$, are necessary to have some sort of "transformation problem" to solve. But as Duménil and Lévy comment on equation (33), "Sequential values are clearly consubstantial with prices, within a *labor-market price* theory of value" (Duménil and Lévy, 2000, p. 127). Equations (33) and (34) show the temporal and logical primacy of observed market prices: the sequence $\{\mathbf{p}_t\}_{t=0,...}$ unidirectionally determines the time paths of all other variables $\{\boldsymbol{\lambda}_t, \mathbf{g}_t, \lambda_{m,t}, \mathbf{p}_t^{TSSI}\}_{t=0,...}$. This is some distance from the classical theory of value.

10. Stochastic Approaches

Despite many conceptual and formal differences, all of the interpretations considered thus far share a common feature: value theory and price theory are analyzed within a *deterministic* framework. In this section, we discuss two less known approaches that substantially deviate from this assumption. They rather conceptualize the main economic magnitudes (prices, values, technology, distribution, and so on) as generated by *stochastic* processes and the transformation problem as relating to *average* values of the relevant variables. In this sense, both approaches adopt what may be defined the *strong average equilibrium price view*. Moreover, the focus of both approaches is on the prices and values of individual commodities and the relation between the two sets of variables at a highly disaggregated level. Thus, they adopt a *microeconomic view*.

10.1 *Stochastic Prices*

From a descriptive perspective, Farjoun and Machover (1983) (henceforth, FM) argue that in general, "labour is, *par excellence*, the essential substance of an economy, and should therefore be taken . . . as the fundamental yardstick. . . . [Economics] *is about the social productive activity of human beings, social labour . . . the study of the social processes and structures by means of which and through which social labour is organized and performed, and the output of this labour distributed and allocated to various uses*" (FM, p. 85). This supports a *predictive view* that labor magnitudes are interpreted probabilistically as the best predictors of actual market monetary magnitudes (prices and profit rates). The fundamental theoretical tenet of their approach is that "the labour theory of value was led into a theoretical crisis not because of the supposed incoherence of the concept of labour-value, nor because it assumed free competition, but because it attempted to reconcile value categories with the fallacious assumption of the uniformity of the rate of profit" (FM, p. 19).

In terms of our axiomatic approach, on the one hand, they take technology as part of the essential data of an economy, and define labor values as the standard input–output employment multipliers, accepting the dualist approach to value magnitudes, including constant and variable capital, and adopting Axiom 2 (**Dualism**). On the other hand, however, they reject Axiom 1 (**Long Period**) and in general any theory of prices based on the assumption of a uniform profit rate. They argue not only that such uniformity is never observed in practice, even as an approximation, but also that "the uniformity assumption is in principle incompatible with a theorization of the capitalist system as a system of free competition and private property in the means of production" (FM, p. 28). For competitive forces constantly tend to create new opportunities for profit and "in a capitalist economy the very forces of competition, *which are internal to the system*, are responsible not only for pulling an abnormally high or low rate of profit back towards normality, *but also for creating such 'abnormal' rates of profit in the first place*" (FM, p. 34). Such competitive forces include not only various pricing and marketing strategies but also technical innovations, which take place at an uneven and uncoordinated pace, and "technical revolutions" (FM, p. 138), which "tend to scramble any putative uniformity in the rate of profit" (FM, p. 35). The uniform profit rate assumption misses the essentially dynamic nature of capitalism.

This entails a different theorization of capitalist economies. First, FM argue that actual market variables should be analyzed adopting "a probabilistic model, in which price, the rate of profit (and other economic parameters, such as capital intensity) are treated from the very beginning not as determinate numerical quantities, but as random variables, each having its own probability distribution" (FM, p. 25). Formally, let K_f and Π_f^{mkt} be, respectively, the total amount of fixed capital (valued at current market prices) owned by firm f and its current profits. The rate of profit of firm f is $r_f = \frac{\Pi_f^{mkt}}{K_f}$. Let w_l be the gross wage paid for the lth worker-hour, and let p_j^{mkt} and λ_j be, respectively, the actual market price paid for a commodity (or a bundle of commodities) in the jth transaction and its labor content. Define $\psi_j \equiv \frac{p_j^{mkt}}{\lambda_j}$: ψ_j is the price paid in the jth transaction per unit of labor content. The first tenet of FM's approach concerns the probabilistic nature of processes generating market outcomes.

Axiom 16 (Stochastic Prices). *Marxian value theory focuses on actual market phenomena and magnitudes. Observed profit rates r_f, wage rates w_l, and prices p_j^{mkt}, are all random variables with given empirical distributions.*

The second key departure from the standard approach concerns the definition of equilibrium. "If a competitive market economy has a state of equilibrium, it must be a state in which a whole range of profit rates coexist; it must be a *dynamic* state, in the sense that the rate of profit of each firm keeps changing all the time; it can only be a state of *equilibrium* in the sense that the *proportion* of capital (out of the total social capital) that yields any particular rate of profit remains approximately constant" (FM, p. 36). This is a statistical equilibrium notion that differs from both the standard Walrasian concept and the long-period approach.

Axiom 17 (Stochastic Equilibrium). *Under perfect competition, the system gravitates toward an equilibrium probability distribution of each random variable, whose general form (at least) is theoretically ascertainable and empirically verifiable.*

Axioms 2, 16, and 17 represent the theoretical core of FM's approach. In order to provide a solution to the transformation problem, however, they need to impose some auxiliary assumptions that allow them to use standard results in probability theory.[31]

First, they postulate that the cumulative density functions of all random variables "can be assumed, with negligible error, to be smooth" (FM, p. 69). Next, using a recursive argument, they show that the price p_j^{mkt} paid for a certain commodity (or bundle of commodities) $\boldsymbol{\chi}$ can be represented as the sum of the total amount of wages, v'_j, paid to all workers who participated directly or indirectly in the production of $\boldsymbol{\chi}$, plus the sum total of profits, s'_j, made by all firms involved directly or indirectly in the production of $\boldsymbol{\chi}$ ("each in respect of its workers' part in the production of this particular commodity" [FM, p. 113]).[32] Therefore, interpreting p_j^{mkt}, v'_j, and s'_j as realizations of three random variables whose relation is captured by the identity $p_j^{mkt} = v'_j + s'_j$, one has

$$\psi_j = \frac{v'_j}{\lambda_j} + \frac{s'_j}{\lambda_j} \tag{41}$$

By equation (41), it follows that the average of ψ is $E\psi = E(\frac{v'}{\lambda}) + E(\frac{s'}{\lambda})$. Using labor values to weight the goods involved in each transaction, by definition, $E(\frac{v'}{\lambda}) = \sum_j \alpha_j \frac{v'_j}{\lambda_j}$ and $E(\frac{s'}{\lambda}) = \sum_j \alpha_j \frac{s'_j}{\lambda_j}$ where $\alpha_j = \frac{\lambda_j}{\sum_j \lambda_j}$. But then at a dynamic equilibrium, $E(\frac{v'}{\lambda}) = \frac{\sum_j v'_j}{\sum_j \lambda_j}$ is the sum total of wages divided by the total amount of labor performed in t, and so $E(\frac{v'}{\lambda}) = Ew$, and we can write $E(\frac{s'}{\lambda}) = e^* E(\frac{v'}{\lambda})$, where $e^* = \frac{\sum_j s'_j}{\sum_j v_j}$ is "the ratio in which the total value-added embodied in the aggregate ... of all commodities sold during [t] is apportioned between profits and wages" (FM, p. 118). Therefore,

$$E\psi = (1 + e^*)Ew \tag{42}$$

This implies that the market prices of commodities are proportional to labor values on average. Furthermore, by the Law of Large Numbers, it follows that if $\boldsymbol{\chi}$ is a large aggregate of commodities sold at total price $p^{mkt}(\boldsymbol{\chi})$ and embodying an amount of labor $\lambda(\boldsymbol{\chi})$, then with high probability and a good level of approximation $\frac{p^{mkt}(\boldsymbol{\chi})}{\lambda(\boldsymbol{\chi})} = E\psi$, and by equation (42)

$$\frac{p^{mkt}(\boldsymbol{\chi})}{\lambda(\boldsymbol{\chi})} = (1 + e^*)Ew \tag{43}$$

Equation (43) holds as an approximation for any large aggregate of commodities, including, for example, the consumption basket of the whole of the working class, **b**.

Next, note that, by definition, the aggregate rate of exploitation is

$$e^M = \frac{L - \lambda(\mathbf{b})}{\lambda(\mathbf{b})} \tag{44}$$

Under the assumption that workers spend all their income, $p^{mkt}(\mathbf{b}) = Ew \cdot L$ and therefore $e^M = \frac{p^{mkt}(\mathbf{b}) - \lambda(\mathbf{b})Ew}{\lambda(\mathbf{b})Ew}$ or, equivalently, by equation (43), with $\boldsymbol{\chi} = \mathbf{b}$,

$$(1 + e^M)Ew = \frac{p^{mkt}(\mathbf{b})}{\lambda(\mathbf{b})} = E\psi$$

which implies that e^M must be equal, or very nearly equal, to e^*.

Finally, let W_f denote the total wage bill paid by firm f and let $z_f = \frac{W_f}{K_f}$ be its organic composition of capital. Define the variable $x_f = \frac{r_f}{z_f}$: x_f is "similar to what Marx calls the *rate of surplus-value*, except that here, too, we measure $[r_f]$ and $[z_f]$ in money terms, whereas he uses labor-values" (FM, p. 69). Let $e_0 = \frac{Er_f}{Ez_f}$: e_0 is the proportion in which the aggregate value added is divided between capital and labor in the economy.

The rates e^* and e_0 are not calculated on the same basis: e_0 "is defined with reference to the *firm space*; if we calculate $[e_0]$, for the period $[t]$, we obtain the ratio in which the *new* value-added *generated during this period* is being shared between capital and labor. On the other hand, $[e^*]$ is defined with reference to the *market space*; it measures the ratio in which the price, which is also the total value-added embodied in [a bundle of commodities $\boldsymbol{\chi}$] – some of which has been generated *before* the period $[t]$ – was shared between capital and labour" (FM, p. 118). Nonetheless, in equilibrium, "the two ratios must be extremely close to each other, because the ratio between total profits and total wages cannot change rapidly" (FM, p. 118). Therefore,

$$E\psi = (1 + e_0)Ew$$

It is now possible to see how these results provide a solution to the transformation problem. Consider Marx's "simple prices" p_j in equation (2). Unlike in FM's framework, they are ideal, rather than market, prices and they are deterministic magnitudes, rather than random variables. If equation (2) holds, then there exists some scalar ψ_0 such that $\frac{p_j}{\lambda_j} = \psi_0$ for all j, including labor power. In Section 3, $\psi_0 = \frac{1}{\lambda_m}$, but FM suggest to normalize prices taking the (ideal) unit wage as the price unit, so that $p_0 = w = 1$, $\psi_0 = \frac{1}{\lambda_0}$, and $\frac{p_j}{\lambda_j} = \frac{1}{\lambda_0}$ for all j. Then, noting that $\lambda(\mathbf{b}) = \lambda_0 L$, equation (44) can be written as

$$e^M = \frac{1 - \lambda_0}{\lambda_0} \tag{45}$$

which, in turn, implies $\frac{1}{\lambda_0} = 1 + e^M$ and therefore,

$$\frac{p_j}{\lambda_j} = 1 + e^M, \quad \text{for all } j \tag{46}$$

Noting that $Ew = w = 1$ by construction, it is possible to see the connection between equation (46), and therefore Marx's "simple prices," p_j, and equations (42) and (43) defining the relation between market prices, p_j^{mkt}, and values in FM's probabilistic approach. Because ψ_j is in general a nondegenerate random variable, equation (46) does not hold in general and individual commodities are unlikely to be exchanged in proportion to their labor values. Nonetheless, under the assumptions of FM's model, equation (43) shows that "when it comes to large and 'unbiased' aggregates of commodities, the specific price of such an aggregate (total price/total labour-content) can, with high probability, be taken as very nearly constant" (FM, p. 135) and the market prices of such aggregates are very close to "simple prices." Equation (42) shows that the same result holds also on average.

Similar conclusions can be reached about the profit rates obtained by capitalist firms on the market. At simple prices, the average rate of profit measured in labor time, r^{FM}, is equal to

$$r^{FM} = \frac{(1-\lambda_o)L}{\lambda(\mathbf{K}_G)} = \frac{L-\lambda(\mathbf{b})}{\lambda(\mathbf{K}_G)} \tag{47}$$

where $\mathbf{K}_G$ is the vector of total capital stocks employed in the economy. Recall that $\mathbf{Q}$ is the gross output vector. Let $\mathbf{I}_G$ denote the vector of intermediate goods used in production: by definition $L = \lambda(\mathbf{Q}) - \lambda(\mathbf{I}_G)$. Then, substituting the latter expression into equation (47) and noting that since $\mathbf{Q}$, $\mathbf{I}_G$, and $\mathbf{K}_G$ are very large aggregates of commodities, equation (43) holds, we can write

$$r^{FM} = \frac{p^{mkt}(\mathbf{Q}) - p^{mkt}(\mathbf{I}_G) - p^{mkt}(\mathbf{b})}{p^{mkt}(\mathbf{K}_G)} \tag{48}$$

But the right-hand side of equation (43) is equal to the average rate of profit, and therefore,

$$r^{FM} = Er \tag{49}$$

In other words, the value rate of profit is equal to the average money rate of profit, proving the link between profits and surplus value.

We can summarize the previous results in the following theorem:

Theorem 8 (The Probabilistic Labor Theory of Value). Under **Dualism, Stochastic Prices**, and **Stochastic Equilibrium**, both the *strong average equilibrium price view* and the *average profit & exploitation view* are logically consistent. Furthermore, **Aggregate Proportionalities** is satisfied.

The approach proposed by FM is innovative and sophisticated. Methodologically, it can be considered as one precursor of the literature on econophysics (Wright, 2005; Cockshott *et at.*, 2009) and of statistical equilibrium theories (Foley, 1994). But it is important to stress that it solves the transformation problem in a very specific and limited sense. The distribution of the random variable $\frac{p_j}{\lambda_j}$ may be rather narrowly clustered around the mean, but it is by no means degenerate. Therefore, commodities do not exchange at labor values—even approximately—when taken individually. Marx's simple prices are a good approximation of market prices only on average (equation (42)), or when large aggregates of commodities are considered (equation (43)), and the probabilistic approach does not (and cannot) provide any explanation of price/value deviations. Nor does it provide any theory of observed market prices, based on labor values or otherwise.

Indeed, labor values play a central role in FM's theorization of the dynamics of capitalist economies, as they capture the deeper technological structure of capitalist production beneath the surface of market phenomena—the real "cost" of goods to society in terms of real human social effort in production. For example, they are arguably the most appropriate measures of labor productivity and can explain long-run effects of technological innovations, including the so-called *law of decreasing labor content* (FM, chapter 7). Yet, in a dynamic perspective, the causality runs from price magnitudes to labor values: actual and expected production costs and profitability determine capitalist innovation activities and choice of techniques, and therefore labor values (see also Flaschel *et al.*, 2013).

One may argue that this lack of theoretical power is compensated by a more realistic set of assumptions and a stronger empirical grounding. For if FM's arguments are correct, then for predictive purposes and from an empirical viewpoint, Marx's simple prices may be taken to be a good approximation of actual market prices. This result, however, is by no means unique to FM's approach. Indeed, a well-known puzzle in the empirical literature on the transformation problem is a very strong correlation between production prices and embodied labor values (see, for example, Shaikh, 2016, chapter 9, and Cockshott and Cottrell, 1997).[33]

10.2 *Stochastic Technology*

The stochastic approach recently proposed by Schefold (2016) shares some important features with FM. Most importantly, like FM, Schefold (2016) adopts the standard definition of values as expressed in Axiom 2 (**Dualism**). However, he differs from FM in two key respects. First, he adopts Axiom 1 (**Long Period**) thereby both endorsing a dualist approach to the definition of values, and defining equilibrium and prices of production in the standard Sraffian fashion. Second, consistent with the adoption of Axiom 1, Schefold rejects Axiom 16 and a focus on market variables. The stochastic nature of the economy emerges from the sphere of production, and not from market processes: it is the fundamental technical data of the economy that should be interpreted as generated by stochastic processes. Formally,[34]

Axiom 18 (Stochastic Technology). *Technology* $(\mathbf{A}, \mathbf{l})$ *is a random variable.*

Although Axioms 1, 2, and 18 represent the theoretical backbone of Schefold (2016)'s approach, the solution to the transformation problem, as with FM, requires some auxiliary assumptions that further specify the properties of the main random variables.

The first assumption concerns the production structure of the economy. Let $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_n)$ be the vector of eigenvalues of the matrix $\mathbf{A}$, where μ_1 is the (strictly positive) dominant eigenvalue.

Assumption 1. *All nondominant eigenvalues of* $\mathbf{A}$ *are (approximately) zero. Formally,* $\mu_i \approx 0$*, for all* $i \neq 1$.

It is well known that $\mu_i = 0$, for all $i \neq 1$ if and only if $\mathbf{A} = \mathbf{cf}$, for some two vectors $\mathbf{c} > \mathbf{0}$, $\mathbf{f} > \mathbf{0}$ where $\mathbf{c}$ is a column vector, $\mathbf{f}$ a row vector, and $\mathbf{A}$ has rank 1. If $\mathbf{A}$ is interpreted as a random matrix that is a perturbation of $\mathbf{A} = \mathbf{c1}$, with $\mathbf{1} \equiv (1, \ldots, 1)$, then the nondominant eigenvalues are only approximately equal to zero. To be precise, suppose the elements of a semipositive and indecomposable matrix $\mathbf{A}$ on each row (i) are distributed independently and identically around a mean specific for the row, and (ii) are random, with a variance that is so large that many single elements equal to zero are admitted. Suppose further that this matrix approximates the form $\mathbf{A} = \mathbf{c1}$, and its dimension is sufficiently large. Then, the nondominant eigenvalues tend to zero, even if the coefficients of $\mathbf{A}$ are perturbed considerably.[35]

In order to state the next assumptions, we need some additional notation. Let $\mathbf{u}_i$ and $\mathbf{i}_i$ denote, respectively, the right (column) and left (row) eigenvectors of $\mathbf{A}$, corresponding to the eigenvalue μ_i, where $\mathbf{u}_1$ and $\mathbf{i}_1$ pertain to the dominant eigenvalue. The components of $\mathbf{u}_1$ are "in the same proportions as Sraffa's standard commodity. This standard vector may also be interpreted as the average industry, introduced by Marx in the third volume of *Das Kapital*" (Schefold, 2016, p. 172). Accordingly, $\mathbf{u}_1$ is called the Sraffa-vector. As for $\mathbf{i}_1$, it is the vector "for which prices would be equal to labor values at all rates of profits, if it were the labour vector" (Schefold, 2016, p. 173), and hence is called the Marx-vector.

The gross output vector and the vector of labor inputs can be expressed as a linear combination, respectively, of the right-hand and the left-hand eigenvectors. Formally, $\mathbf{Q} = \sum_{i=1}^{n} \mathbf{u}_i$ and $\mathbf{l} = \sum_{i=1}^{n} \mathbf{i}_i$. The next assumption then imposes a constraint on the sectoral deviations of the gross output vectors and the labor vector from the Sraffa-vector and the Marx-vector: $\mathbf{d^u} = \mathbf{Q} - \mathbf{u}_1$ and $\mathbf{d^i} = \mathbf{l} - \mathbf{i}_1$.

Assumption 2. *The deviations of activities from the average industry and the deviations of the labor vector from the Marx-vector are not correlated. Formally,* $cov(\mathbf{d^u}, \mathbf{d^i}) = 0$.

The next assumption focuses on some properties of surplus products and the labor vector.

Assumption 3. *The deviations of the labor vector from the Marx-vector and the vector* $\mathbf{s}$ *of surplus products are not correlated. Formally,* $cov(\mathbf{s}, \mathbf{d^i}) = 0$.

If the input matrix is a perturbation of $\mathbf{A} = \mathbf{cf}$, for a generic $\mathbf{f} > \mathbf{0}$, then one more, crucial assumption is necessary, which concerns the average of the deviations of the labor vector from the Marx-vector. For every vector $\boldsymbol{\gamma}$, let $\overline{\gamma}$ denote the average of the components of $\boldsymbol{\gamma}$.

Assumption 4. *On average, the deviations of the labor vector from the Marx-vector disappear. Formally,* $\overline{d}^i = 0$.

According to Schefold (2016, p. 174), Assumption 4 "means that, because the individual deviations of the labour vector from the Marx-vector do not disappear but its average disappears, the labour theory of value does not hold for the single prices but on average, as it were." Formally, Assumptions 2 and 3 imply, respectively, that $\mathbf{d^i d^u} = n\overline{d}^i\overline{d}^u$ and $\mathbf{d^i s} = n\overline{d}^i\overline{s}$, and by Assumption 4, the latter vector products are both equal to zero. If, however, the input matrix is a perturbation of $\mathbf{A} = \mathbf{c1}$, then Assumption 4 follows as a *result* as proved in the theory of stochastic matrices. Either way, this is the key property to prove the following theorem.[36]

Theorem 9 (The Average Labor Theory of Value). Under **Assumptions 1–4, Dualism, Long Period**, and **Stochastic Technology**, both the *strong average equilibrium price view* and the *average profit & exploitation view* are logically consistent. Furthermore, **Aggregate Equalities** is satisfied.

According to Schefold (2016, p. 176), Theorem 9 is "a most surprising result, obtained after 120 years of discussions of the transformation problem." It establishes that "the Marxian transformation of values into prices is correct after all, despite many refutations, if the economic system under consideration is random" (Schefold, 2016, p. 165). Theorem 9 is indeed a remarkable result as it provides a solution to the transformation problem within the standard dualist framework, due to an innovative interpretation in terms of random matrices. Furthermore, Schefold (2016, section 3) relates some key aspects of the formalism to Marx's texts, providing an interesting interpretation of dialectics.

Nonetheless, two caveats should be made concerning the interpretation of the results. First, "Prices are here not derived from values, but without having recourse to values from the structure of production or of the values in use, represented by $\mathbf{A}$ and $\mathbf{l}$, and from the distribution, represented by r. The formal redundancy of the theory of surplus value remains" (Schefold, 2016, p. 177).

Second, Schefold criticizes and rejects the NI because some of its results are "little more than a tautology" (Schefold, 2016, p. 170). Yet, from a logical perspective, Schefold's approach is very similar in that the key Marxian insights, and Theorem 9, follow straightforwardly by virtue of the axioms and definitions. This is not to suggest that this approach (or others) is trivial. Rather it emphasizes the fact that in *all* approaches, the results follow in some sense from the relevant definitions and from the axiomatic framework characterizing a given approach.

Indeed, our axiomatic treatment very clearly suggests that the strength of Theorem 9 lies entirely in the strength of the underlying axioms, and the axioms are not entirely convincing, or at least are insufficiently motivated. For example, concerning Assumption 2 Schefold (2016, p. 173) simply says: "Now there is in fact no reason why the deviations of activities from the average industry and the deviations of the labour vector from the Marx-vector should be correlated." Yet there is no reason why they should not be correlated (or at least no independent reason is provided). Even more puzzlingly, concerning Assumption 4, he simply says "that on average the deviations of the labour vector from the Marx-vector disappear. This is a new assumption" (Schefold, 2016, p. 174). To be sure, as mentioned earlier, Assumption 4 can be obtained as a result from more basic premises, namely, the assumption that $\mathbf{A}$ is a stochastic matrix that can be approximately seen as the perturbation of $\mathbf{A} = \mathbf{c1}$. Yet, it is unclear why—either theoretically or empirically—the matrix of material input requirements should be even approximately of rank 1, let alone have essentially identical columns. At a deeper level, one may even question the assumption that the technology matrix $\mathbf{A}$ can be meaningfully considered to be random in the statisticians' sense of the word, as it is the outcome of certain processes of innovation and choice of techniques.[37]

11. Conclusions

This paper provides a new interpretation of the literature on the transformation problem by using the language of modern axiomatic theory. This approach has significant advantages in terms of clarifying the exact nature and scope of the argument. On the one hand, it allows us to show that, on its own terms, the transformation problem is an impossibility result. At a purely logical level, there is nothing to discuss about it and there is no hope of "solving" it. On the other hand, however, it forcefully shows that the result depends both on a certain interpretation of Marxian value theory *and* on a specific set of assumptions and definitions—a specific axiomatic structure.

In the standard dualist approach that has dominated the debate from the publication of *Capital III* up until the 1970s, money plays no role and labor values and monetary magnitudes are assumed to form two conceptually separate systems. There is an underlying (intrinsic, invisible, and essential) system of labor values and associated exploitation, and a phenomenal (extrinsic, visible, and superficial) system of prices and profit rate. Marxian value theory is then interpreted as a predictive tool that bridges the gap between the two systems: relative labor values are meant to explain equilibrium relative prices. Because no robust relation between labor and monetary magnitudes can be proved in the dualist framework, the conclusion is that Marxian value theory is at best irrelevant, if not irremediably inconsistent.

The axiomatic approach adopted in this paper has the advantage of clarifying the key assumptions and the logical structure of the received approach, and it has allowed us to show that neither its general conception of value theory, nor the specific axioms adopted are logical truths. Both can be, and indeed have been, modified in various logically consistent and theoretically relevant directions. In closing this paper, it is worth summarizing what we believe are the key departures from the standard view of these recent approaches (albeit, as we have shown, with different emphases).

First, a strictly predictive interpretation of Marxian value theory is unnecessarily reductive. The labor theory of value can be meaningfully interpreted as a tool for describing and understanding the basic structure and dynamics of capitalist economies, and in particular the relation between profits and exploitation, even if embodied labor values are not good theoretical predictors of prices of production.

Second, dimensionality is important. The standard approach is dualist in that it interprets labor accounts and monetary magnitudes as unrelated and separate systems. Its focus on *relative* values and prices eliminates and therefore obscures the fact that these magnitudes are denominated in different units. Once that difference is kept explicit, then the translation between the two relates labor magnitudes and monetary magnitudes in a way that the dualist approach does not manage.

This leads immediately to the third feature: the labor theory of value is a monetary theory (because a capitalist economy is a monetary economy) and that must require a conversion rate to move back and forth from labor accounts to money accounts. So the value of money is a central concept. But more than that, money is not a veil, concealing a set of real transactions; it is rather how value appears when it is separated from the commodity. This further implies that the wage transaction is a monetary one: the sale of labor power is for a monetary wage, and the value of labor power is that wage multiplied by the value of money.

No survey is theoretically innocent. We began in Section 2 by noting a number of possible interpretations of the labor theory of value. At its most general and abstract level, the labor theory of value is a statement that as long as labor is mobile, a decentralized allocation of labor is organized via the natural prices of all activities when these are proportional to the human effort expended in such activities. This allocation of labor is modified by the distribution via class relations of the monetary form of the surplus product. The theoretical challenge is to understand how this modification works in a context of class exploitation. While this motivates the NI in particular, the various approaches we have surveyed all have something that underpins their different assumptions/axioms. This paper has surveyed the logic of the latter, but it should be clear that that logic can only take us so far.

Our survey then does not provide the final word on Marxian value theory. Among other things, we have developed our analysis at a purely theoretical level, and have neglected the important issue of its empirical relevance. We hope to have shown, however, that the central question is not whether the transformation problem can be solved, but rather whether modern approaches to value theory can provide a theoretically rigorous framework that can underpin the analysis of contemporary capitalist economies.

Notes

1. There is no single, unequivocal way of defining an approach axiomatically. It is largely a matter of emphasis whether certain features should be considered as part of the specification of the domain of problems, or rather of the desirable properties of solutions for the domain. Different axiomatic descriptions may be appropriate depending on the theoretical exercise. The axiomatic method is not "a substitute for intuition ... but instead ... a way of articulating [the intuitions that hold in specific situations] into operationally useful conditions pertaining to an entire class of cases" (Thomson, 2001, p. 356).
2. Although the discussion in Section 5 touches upon various issues that are central in the Sraffian literature, the latter falls beyond the scope of our paper. For Sraffians, the labor theory of value has no relevance, and the issues that were central to Ricardo and Marx can be analyzed with Sraffa's long-period framework. For a discussion, see Steedman (1977) and Mongiovi (2002).
3. This can be interpreted both as a descriptive and as a normative claim. As such, it offers one possible reason why we focus on a labor theory of value rather than, say, on a "steel theory of value" and the corresponding transformation from "steel values" at simple prices to prices of production.
4. These simplifying assumptions are made for expositional purposes, since the transformation problem arises, and has traditionally been discussed, within the simplified context analyzed here. It is worth stressing, however, that modern approaches to Marxian price and value theory provide solutions to the transformation problem that are independent of these simplifying assumptions. See, for example, Flaschel (1983, 2010) and Flaschel *et al.* (2013) on joint production and fixed capital; Duménil *et al.* (2009) and Veneziani and Yoshihara (2015, 2017) on heterogeneous labor.
5. These refinements of labor and labor time are not the subject of this paper. Henceforth, the qualifier "socially necessary" will be dropped; but units of time in this paper are assumed to be always so qualified.
6. In this paper, we follow Marx and assume, without loss of generality, that wages are paid *ex ante*.
7. The inverse of λ_m is denominated in units of money per unit of time, and is accordingly called "the monetary equivalent of labor time." While some authors work with this directly, we use the value of money throughout.
8. In his numerical examples, Marx was generally explicit about this; for example, he wrote, "if 2 ounces of gold when coined are £2 ... " (Marx, 1976, p. 163), and "If then, twenty-four hours of labour, or two working days, are required to produce the quantity of gold represented by 12 shillings ... " (Marx, 1976, p. 294). For an analysis of Marx's concept of money, see Foley (1986). We do not pursue the philosophical and exegetical literature on money in this paper. A sample might include Nelson (1999) and Lapavitsas (2017).
9. The assumption that workers do not save is conceptually central to the dualist approach. For expositional clarity we maintain it throughout the paper to facilitate the comparison of alternative approaches. Yet none of the key arguments of the paper depends on ruling out workers' savings.
10. We assume throughout that hours of labor power hired are unproblematically translated into hours of labor worked.

11. We assume that **A** is nonnegative, productive, and indecomposable, and **l** is strictly positive.
12. These quantities are all determined *ex post*; they are givens of the analysis, and no assumption concerning constant returns to scale is made (nor is it necessary, Flaschel, 2010).
13. Given our assumptions on technology $(\mathbf{A}, \mathbf{l})$, it is well known that, due to a theorem of Frobenius, Perron, and Remak, for a range of values of the distributive variables, once either of (r^e, w^e) is specified, equation (11) has a unique, economically meaningful solution. See, for example, Roemer (1981) and Flaschel (2010).
14. By equation (16), the (equalized) rate of profit in equation (12) becomes $r^e = \frac{\Pi^e}{\mathbf{p}^e\mathbf{AQ}+\mathbf{p}^e\mathbf{b}}$.
15. We prefer "total revenue" to the terminologically imprecise "total price" that is typically used in the literature.
16. A third, more recent solution within the dualist approach is examined in Section 10.2 below.
17. Dimensionality issues are not relevant here since the price vector is the eigenvector of the augmented input coefficients matrix **M**, and is determined up to a positive multiplicative constant, so that only relative prices matter.
18. According to Shaikh, (2016, chapter 6) and the references therein, the reason why part (ii) fails to hold is because of transfers between the "circuit of capital" and the "circuit of revenue."
19. See also Lipietz (1982), Mohun (1994, 2004), Duménil and Foley (2008), and Foley and Mohun (2016). It is worth noting that other authors had anticipated *some* of the key elements of the NI such as the definition of the value of labor power (Robinson, 1965; Schefold, 1973) or the existence of a conversion rate between money and labor accounts (Desai, 1979). However, Duménil and Foley were the first to put these elements together into a unified approach.
20. The terminology "commodity law of exchange" to describe equation (2), and "capitalist law of exchange" to describe the determination of prices that support an equalized rate of profit, is used by Foley and Mohun (2016).
21. As already noted, the adoption of the *macroeconomic view* does not imply that microeconomic variables are ignored, or irrelevant in the NI. Indeed, Duménil *et al.* (2009) have extended the NI at the meso-level by analyzing value creation at the sectoral level. Further, in a series of recent contributions, Veneziani and Yoshihara (2015), Veneziani and Yoshihara (2017), and Yoshihara (2010) have shown axiomatically that the NI can provide an appropriate criterion to analyze individual exploitation status.
22. Roberts (1997, 2009) defends a strong version of the *profit & exploitation view* that holds at the level of individual industries or processes.
23. Of course, only relative prices of production are thereby determined; the further normalization equation (26) is discussed below.
24. Furthermore, Axiom 1 so reinterpreted is not understood as describing the long-period position of the economy but as representing, for Marxism, the "condition for equivalent exchange" under the competitive capitalist conditions of *Capital III*, Part I. We are grateful to Bruce Roberts for this suggestion.
25. Alternatively, instead of Axiom 8, equation (26) could be given axiomatic status with equation (24) derived as a result. This choice makes no difference for our conclusions. We do think, however, that Axiom 8 reflects Wolff *et al.* (1982, 1984a, b)'s own presentation of the approach.
26. Roberts (1997, 2009) has developed the WCR approach by further analyzing the relation between prices and values, and by considering economies with joint production and heterogeneous labor. Yet, the fundamental axiomatic structure of the approach is the same and it remains true that prices are defined independently of values but the converse is not true.
27. The part of claim (d) concerning the role of values in the *determination* of prices is not entirely clear and we shall return to it later.

28. In Theorem 7, TSSI proponents also include: "(g) labor-saving technical change itself can cause the profit rate to fall" (Kliman and McGlone, 1999, p. 55). But claim (g) cannot be proved based only on Axioms 11–15. Indeed, the asserted TSSI relation between labor-saving innovations and movements in the rate of profit is controversial (for example, Veneziani, 2004). Different interpretations of (price and) value theory certainly have implications for the analysis of the dynamics of a capitalist economy, but space constraints preclude their exploration in this paper.
29. The following discussion draws heavily on Veneziani (2004).
30. Sometimes, TSSI proponents suggest that the definition of $\lambda_{m,t}$ can be derived by postmultiplying equation (34) by $\mathbf{Q}_{t+1}$, and rearranging to obtain

$$\frac{1}{\lambda_{m,t+1}} = \frac{\mathbf{p}_{t+1}\mathbf{Q}_{t+1}}{\lambda_{m,t}\mathbf{p}_t\mathbf{A}_t\mathbf{Q}_{t+1} + \mathbf{l}_t\mathbf{Q}_{t+1}}$$

Unfortunately, this equation does not provide a *definition* of $\lambda_{m,t+1}$: it describes its motion, provided that $\lambda_{m,0}$ is independently defined. And there is no such definition in the TSSI literature. For further discussion, see Mohun and Veneziani (2009).
31. See Fröhlich (2012) for an econometric analysis of the basic assumptions of FM.
32. Therefore, v'_j and s'_j are different from v_i and s_i used, for example, in equation (6): they are monetary (not value) magnitudes and capture the interconnectedness of the economic system. Thus, v'_j measures the wages paid to the workers involved in the production of the goods in transaction j, plus the wages paid in the production of the intermediate goods necessary to produce such goods, and so on.
33. But note that the input–output methodology of these empirical studies implicitly assumes all labor to be productive, which raises some delicate issues in interpreting such correlations as having anything to do with values.
34. "Essentially, matrices are random, if the elements on each row (which represents the process) are i.i.d. with a distribution around a mean specific for the row" (Schefold, 2016, p. 166).
35. "Tend to zero" here means, as usual, that the modulus of any eigenvalue is smaller than any preassigned positive number. For a more thorough discussion of the relevant assumptions, see Schefold (2013).
36. Observe that Theorem 9 proves that **Aggregate Equalities** holds. This is because, in Schefold's framework, only relative prices matter. The proof of Theorem 9 is in the Addendum (in the Additional Supporting Information in the online version of this article).
37. We are grateful to Gary Mongiovi for this suggestion.

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